QUANTITATIVE QUASISYMMETRIC UNIFORMIZATION OF COMPACT SURFACES

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Abstract. Bonk and Kleiner showed that any metric sphere which is Ahlfors 2-regular and linearly locally contractible is quasisymmetrically equivalent to the standard sphere, in a quantitative way. We extend this result to arbitrary metric compact orientable surfaces.

1. Introduction and statement of results

Through isothermal coordinates, every Riemannian metric on a compact orientable surface S determines a Riemann surface structure on S. By the classical uniformization theorem, S carries a conformally equivalent Riemannian metric of constant curvature 1 (in the case of a sphere), 0 (for a torus), or -1 (for higher genus surfaces). Hence, the original Riemannian metric on S can be *conformally deformed* to a metric of constant curvature.

The purpose of this note is to extend the above discussion to certain classes of possibly non-smooth distances on compact orientable surfaces. In this setting we have a metric, but no smooth structure, and the appropriate category replacing the class of conformal mappings is the class of *quasisymmetric mappings.*

In a metric space X we will denote the distance between a and b by $|a-b|_X$, or $|a-b|$ if the metric space X is clear from the context. Let X and Y be metric spaces. An embedding $\phi\colon X \to Y$, i.e., a homeomorphism from X onto its image $\phi(X)$, is a *quasisymmetric embedding* if there is a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that for all triples of distinct points a, b, and c in X,

$$\frac{|\phi(a)-\phi(b)|_Y}{|\phi(a)-\phi(c)|_Y} \leq \eta\left(\frac{|a-b|_X}{|a-c|_X}\right),$$

The homeomorphism η is called a *distortion function* for the mapping ϕ. If a quasisymmetric embedding ϕ has a distortion function η, it will be called an η-quasisymmetric embedding. The inverse of an η-quasisymmetric map is η'-quasisymmetric with $\eta'(t) = 1/\eta^{-1}(1/t)$. A proof of this fact can be found in the book of Heinonen [Hei01, Proposition 10.6], which also serves as an excellent introduction to the theory of quasisymmetric mappings on

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metric spaces. A quasisymmetric mapping distorts *relative* distances by a controlled amount. While it may distort distances, it must do so more-or-less isotropically.

A motivating example arises from Cannon's Conjecture in geometric group theory. The boundary at infinity of a Gromov hyperbolic group carries a natural family of visual metrics, any two of which are quasisymmetrically equivalent. For this reason, one might say that the metric on the boundary of a Gromov hyperbolic group is defined only up to quasisymmetry. Cannon's Conjecture can be phrased as follows: If the boundary $\partial_\infty G$ of a Gromov hyperbolic group G is homeomorphic to the sphere $\mathbb{S}^2$, then each visual metric on $\partial_\infty G$ is quasisymmetrically equivalent to the standard metric on $\mathbb{S}^2$. If Cannon's conjecture is true, then a Gromov hyperbolic group whose boundary is homeomorphic to $\mathbb{S}^2$ acts discretely and co-compactly by isometries on 3-dimensional hyperbolic space. See [Bon06] and the references therein for a more detailed discussion of this problem.

The following quasisymmetric uniformization theorem of Bonk and Kleiner [BK02] led to significant progress on Cannon's Conjecture:

Theorem 1 (Bonk-Kleiner)**.** *Let (X,d) be an Ahlfors 2-regular metric space that is homeomorphic to $\mathbb{S}^2$. Then (X,d) is quasisymmetrically equivalent to $\mathbb{S}^2$ equipped with the smooth metric of constant curvature 1 if and only if (X,d) is linearly locally contractible.*

A metric space X is *linearly locally contractible*, if there is a constant $\Lambda \geq 1$ such that for each point $x \in X$ and radius $0 < r < (\operatorname{diam} X)/\Lambda$, the ball $B(x,r)$ is contractible inside the dilated ball $B(x,\Lambda r)$. The metric space X is *Ahlfors 2-regular* if there is a constant $K \geq 1$ such that for each $x \in X$ and radius $0 < r < 2 \operatorname{diam} X$, the two-dimensional Hausdorff measure $\mathcal{H}^2$ satisfies

$$\frac{r^2}{K} \leq \mathcal{H}^2(B(x,r)) \leq K r^2. \tag{1}$$

Given a Gromov-hyperbolic group G with $\partial_\infty G$ homeomorphic to $\mathbb{S}^2$, each visual metric on $\partial_\infty G$ is linearly locally contractible. However, it is not known whether there always is an Ahlfors 2-regular visual metric on $\partial_\infty G$. Simple examples show that Theorem 1 fails without the assumption of Ahlfors 2-regularity, although Ahlfors 2-regularity is not preserved by quasisymmetric mappings in general.

The linear local contractibility condition is a quantitative quasisymmetric invariant, i.e., if $f\colon X \to Y$ is η-quasisymmetric and X is linearly locally contractible with constant Λ, then Y is linearly locally contractible with a constant that depends only on Λ and η.

Theorem 1 is quantitative in the sense that the quasisymmetric map can be chosen to have a distortion depending only on the *data* of X, i.e., the constants Λ and K appearing in the linear local contractibility and Ahlfors 2-regularity conditions.

Problems outside of geometric group theory, such as the search for an intrinsic characterization of $\mathbb{R}^2$ up to bi-Lipschitz equivalence (note that Ahlfors 2-regularity *is* a bi-Lipschitz invariant), led to the development of versions of Theorem 1 for non-compact simply connected surfaces [Wil08] and for a large class of planar domains [MW13], as well as a local version [Wil10]. In this work, we provide a version of Theorem 1 that applies to all orientable compact surfaces, including those of higher genus. This is part of an ongoing program to complete the classification of Ahlfors 2-regular and linearly locally contractible metric surfaces up to quasisymmetry.

Theorem 2. *Let (X,d) be a metric compact orientable surface. Assume that (X,d) is Ahlfors 2-regular and linearly locally contractible. Then there exists a Riemannian metric $\hat{d}$ of constant curvature 1, 0, or -1 on X such that the identity map $\mathrm{id}:(X,d)\to(X,\hat{d})$ is quasisymmetric with a distortion function depending only on the data of X.*

Here the *data* of X are the constants in the Ahlfors 2-regularity and linear local contractibility conditions.

Except for the statement of dependence of the distortion function only on the data of (X,d), Theorem 2 follows immediately from the local uniformization theorem of [Wil10] and an elementary local-to-global result for quasisymmetric mappings due to Tukia and Väisälä [TV80, Theorem 2.23]. A quantitative proof is significantly more involved.

We note that the data of (X,d) do *not* measure the "quasisymmetric distance" of (X,d) to a Riemannian metric of constant curvature. For each genus $g\leq 0$ and any $\Lambda\geq 1$, there is a compact Riemannian surface of genus g and constant curvature that fails to be Λ-linearly locally contractible; this occurs when the injectivity radius is a sufficiently small compared to the diameter. Note also that as the genus grows, the ratio of the injectivity radius to the diameter necessarily tends to 0.

This indicates a very interesting, though vague, question: given (X,d) as in Theorem 2, find a the Riemannian metric $\hat{d}$ on X and a quasisymmetric mapping $f\colon(X,d)\to(X,\hat{d})$ with "minimal" quasisymmetric distortion over all such metrics and mappings.

0ne might hope to improve Theorem 2 by separating the data into parts which control the *scale* and *severity* of the conditions separately. Suppose that (X,d) is a metric compact orientable surface so that there are constants $C_1,C_2\geq 1$ so that for all $x\in X$ and $0<r<(\operatorname{diam}X)/C_1$, the ball $B(x,r)$ is contractible inside $B(x,C_2r)$ and satisfies (1) with K replaced by C_2. Unfortunately, Theorem 2 does not hold if the data of X is defined to be the single number C_2; a counter-example is provided by a sequence of smooth dumbell-shaped spheres with narrowing "bars"; by allowing C_1 to grow, we may choose C_2 to be constant for the entire sequence. Despite this, the quasisymmetric distortion of mappings to a sphere of constant curvature must grow. A similar construction can be made in any genus.

Our proof of Theorem 2 proceeds as follows. Let (X, d) be a metric surface as given in the theorem. By the local uniformization result of [Wil10], (X, d) has an atlas of uniformly quasisymmetric mappings. Our first step is to create a compatible conformal atlas, giving (X, d) the structure of a Riemann surface. This step can be thought of as creating "quasi-isothermal" coordinates; while the chart transitions are conformal, the chart mappings themselves are only quasisymmetries. The classical uniformization theorem then provides a globally defined conformal homeomorphism $F\colon X \to Y$ to a Riemann surface that is the quotient of the sphere, plane, or disk by an appropriate group of Möbius transformations. The Riemann surface Y inherits a Riemannian metric d_Y of constant curvature 1, 0, or -1, respectively. Although conformal maps are locally quasisymmetric, this does not immediately give us global information about the metric properties of the uniformizing map $F\colon (X, d) \to (Y, d_Y)$. We show that the uniformizing mapping F is in fact globally quasisymmetric with a distortion function that depends only on the data of X. The key idea in this step is a type of Harnack inequality. The claim in the theorem then follows by taking $\hat{d}$ to be the pull-back of the metric d_Y under F.

2. Background and preliminary results

2.1. Notation.

In a metric space X, we will denote the distance between points x and y of X by $|x - y|_X$ or $|x - y|$ when the space X is understood. We denote by

$$B_X(x, r) = B(x, r) = \{y : |x - y| < r\}$$

the open ball of radius r centered at x and by $\overline{B}(x, r)$ the corresponding closed ball. For an open or closed ball B of radius r and a number $\lambda > 0$, the notation λB denotes the same type of ball with the same center and radius λr.

We denote the complex plane with the standard Euclidean metric by $\mathbb{C}$, and the disk model of hyperbolic space equipped with the standard hyperbolic metric of curvature -1 by $\mathbb{D}$. The open ball of radius $\alpha > 0$ centered at 0 is denoted by $\mathbb{C}_\alpha$ in $\mathbb{C}$ and by $\mathbb{D}_\alpha$ in $\mathbb{D}$, i.e.,

$$\mathbb{C}_\alpha = B_{\mathbb{C}}(0, \alpha), \text{ and } \mathbb{D}_\alpha = B_{\mathbb{D}}(0, \alpha).$$

Note that α is the Euclidean radius for $\mathbb{C}_\alpha$, whereas it is the hyperbolic radius for $\mathbb{D}_\alpha$.

2.2. **Quasisymmetric mappings.** Let $\phi\colon X \to Y$ be an embedding of metric spaces. For $x \in X$ and $0 < r < \frac{1}{2}\operatorname{diam} X$, define

$$L_\phi(x,r) := \sup\{|\phi(x) - \phi(y)|_Y : |x-y|_X \leq r\}, \text{ and}$$
$$l_\phi(x,r) := \inf\{|\phi(x) - \phi(y)|_Y : |x-y|_X \geq r\}.$$

If ϕ is an η-quasisymmetric embedding, then for $x \in X$ and $0 < r_1, r_2 < \frac{1}{2}\operatorname{diam} X$,

$$\frac{L_\phi(x,r_1)}{l_\phi(x,r_2)} \leq \eta\left(\frac{r_1}{r_2}\right).$$

We will need a statement about the equicontinuity of quasisymmetric maps, which is a slightly more general version of [Hei01, Corollary 10.27].

Lemma 3. *Let $\phi\colon U \to V$ be an η-quasisymmetric embedding of metric spaces and let D_U and D_V be positive real numbers. If $\operatorname{diam} U \geq D_U$ and $\operatorname{diam} V \leq D_V$, then*

$$\omega(t) = \max\left\{\frac{3D_V t}{D_U}, D_V\eta\left(\frac{3t}{D_U}\right)\right\}$$

is a modulus of continuity for ϕ.

Proof. Given $x_1, x_2 \in U$ with $0 < |x_1 - x_2| < D_U/3$, there exists $x_3 \in U$ such that $|x_1 - x_3| \geq D_U/3$. Then

$$|\phi(x_1) - \phi(x_2)| \leq |\phi(x_1) - \phi(x_3)|\eta\left(\frac{|x_1 - x_2|}{|x_1 - x_3|}\right) \leq D_V\eta\left(\frac{3|x_1 - x_2|}{D_U}\right).$$

If $|x_1 - x_2| \geq D_U/3$, the trivial estimate $|\phi(x_1) - \phi(x_2)| \leq D_V$ yields the desired estimate. □

Note that the modulus of continuity provided by Lemma 3 is not scale-invariant. If the metrics on U and V are scaled by the same quantity, the resulting modulus of continuity may still change; see subsection 2.4.

The following result, which is a slight variation of [TV80, Theorem 2.23], gives a local-to-global result for quasisymmetric homeomorphisms between compact spaces in terms of Lebesgue numbers. We include a proof for the reader's convenience. Recall that $L > 0$ is a *Lebesgue number* for a covering $\{X_j\}$ if for every set E with $\operatorname{diam} E \leq L$ there exists j such that $E \subseteq X_j$.

Theorem 4 (Tukia-Väisälä)**.** *Let $F\colon X \to Y$ be a homeomorphism between compact connected metric spaces X and Y. Suppose that*

- *$\{X_j\}_{j=1}^n$ is a finite open covering of X with Lebesgue number $L_X < \operatorname{diam} X$.*
- *$\delta > 0$ satisfies the implication*

$$|x - x'| = L_X/2 \implies |F(x) - F(x')| \geq \delta,$$

- *there is a quasisymmetric distortion function η such that for each $j = 1, \dots, n$, the restricted mapping $F|_{X_j}$ is η-quasisymmetric.*

Then F is quasisymmetric with distortion function depending only on η and the ratios $\operatorname{diam}(X)/L_X$ *and* $\operatorname{diam}(Y)/\delta$*.*

Proof. Let a, b, and c be distinct points of X. Set

$$\rho = \frac{|a-b|}{|a-c|}, \text{ and } \rho' = \frac{|F(a)-F(b)|}{|F(a)-F(c)|}.$$

We consider four cases.

(1) Suppose that $\max\{|a-b|, |a-c|\} \le L_X/2$. Then $\{a,b,c\}$ is contained in some X_j, and so $\rho' \le \eta(\rho)$.

(2) Suppose that $|a-b| \le L_X/2$ but $|a-c| > L_X/2$. Since X is connected with $\operatorname{diam} X > L_X$, there exists $x \in X$ with $|a-x| = L_X/2$. Then there exists j such that $\{a,b,x\} \subseteq X_j$ and

$$|F(a)-F(x)| \ge \delta \text{ and } \frac{|a-b|}{|a-x|} \le \frac{2\rho \operatorname{diam} X}{L_X}.$$

This implies that

$$\rho' = \frac{|F(a)-F(b)|}{|F(a)-F(x)|}\frac{|F(a)-F(x)|}{|F(a)-F(c)|} \le \eta\left(\frac{2\rho \operatorname{diam} X}{L_X}\right)\frac{\operatorname{diam} Y}{\delta}.$$

(3) Suppose that $|a-b| > L_X/2$ but $|a-c| \le L_X/2$. By the same argument as in the previous case, there exists $x \in X$ with $|a-x| = L_X/2$, as well as an index j such that $\{a,c,x\} \subseteq X_j$. This gives

$$|F(a)-F(x)| \ge \delta \text{ and } \frac{|a-x|}{|a-c|} \le \frac{2\rho \operatorname{diam} X}{L_X}.$$

which implies that

$$\rho' = \frac{|F(a)-F(b)|}{|F(a)-F(x)|}\frac{|F(a)-F(x)|}{|F(a)-F(c)|} \le \frac{\operatorname{diam} Y}{\delta}\eta\left(\frac{2\rho \operatorname{diam} X}{L_X}\right).$$

(4) Suppose that $\min\{|a-b|, |a-c|\} > L_X/2$. Then $\rho \ge \frac{L_X}{2 \operatorname{diam} X}$ and $\rho' \le \frac{\operatorname{diam} Y}{\delta}$, so

$$\rho' \le 2\left(\frac{\operatorname{diam} X}{L_X}\right)\left(\frac{\operatorname{diam} Y}{\delta}\right)\rho$$

Combining these estimates on ρ' in terms of ρ yields the desired result. □

2.3. **Conformality and quasisymmetry.** A conformal mapping $f\colon U \to V$ between domains in $\mathbb{C}$ is quasisymmetric when restricted to a relatively compact subdomain $U' \subset U$, with quasisymmetric distortion depending only on U and U'. This fact, which should be compared to Koebe's distortion theorem, will play a key role in the proof of Theorem 12. A more general statement is even true: one may consider quasiconformal mappings in higher dimensional Euclidean spaces, although the quasisymmetric distortion then also depends on the maximal quasiconformal dilatation. See [Väi81, Theorem 2.4] for a proof of this fact, which we will use in the following form:

Proposition 5. *Let $G\colon \mathbb{C}_1 \to \mathbb{C}$ be a conformal embedding. For each $\beta \in (0,1)$, the restriction $G\colon \mathbb{C}_\beta \to \mathbb{C}$ is a quasisymmetric embedding with distortion function that depends only on β.*

An easy consequence of Proposition 5 is the following.

Corollary 6. *There is a universal constant $0 < \beta_1 < 1/2$ such that for any conformal embedding $G\colon \mathbb{C}_1 \to \mathbb{C}$,*

$$G(\mathbb{C}_{\beta_1}) \subseteq B_{\mathbb{C}}\left(G(0), \frac{l_G(0,1/2)}{6}\right).$$

Proof. By Proposition 5, there is a universal quasisymmetric distortion function η for $G\colon \mathbb{C}_{1/2} \to \mathbb{C}$. Hence, if $\beta \in (0,1/2)$, then

$$L_G(0,\beta) \leq \eta\,(2\beta)\, l_G(0,1/2)$$

Thus, choosing $0 < \beta_1 < 1/2$ so small that $\eta(2\beta_1) < 1/6$ fulfills the requirements of the statement. □

We will also need a hyperbolic version of this result. Recall that the hyperbolic disk $\mathbb{D}$ is equipped not just with a conformal structure, but also with the standard hyperbolic metric. Our proof employs Koebe's distortion theorem for specificity, but could also be carried out using Proposition 5 alone.

Proposition 7. *Let $G\colon \mathbb{C}_1 \to \mathbb{D}$ be a conformal embedding of the Euclidean unit disk into the hyperbolic plane. Then the restriction $G\colon \mathbb{C}_{1/10} \to \mathbb{D}$ is a quasisymmetric embedding with a universal distortion function.*

Proof. The map $f(z) = \frac{G(z)-G(0)}{G'(0)} = z + O(z^2)$ is conformal in $\mathbb{C}_1$, so by Koebe's distortion theorem [Dur83], the image $f(\mathbb{C}_1)$ contains the Euclidean disk $\mathbb{C}_{1/4}$, and the image of the Euclidean disk $\mathbb{C}_{1/10}$ is contained in the Euclidean disk $\mathbb{C}_{10/81} \subseteq \mathbb{C}_{1/8}$. This implies

$$B_{\mathbb{C}}\left(G(0), \frac{|G'(0)|}{4}\right) \subseteq G(\mathbb{C}_1) \subseteq \mathbb{D},$$

and

$$G\left(\mathbb{C}_{1/10}\right) \subseteq B_{\mathbb{C}}\left(G(0), \frac{|G'(0)|}{8}\right) \subseteq B_{\mathbb{C}}\left(G(0), \frac{1-|G(0)|}{2}\right).$$

It follows from the definition of the hyperbolic metric that the identity mapping

$$\mathrm{id}\colon B_{\mathbb{C}}\left(G(0), \frac{1-|G(0)|}{2}\right) \to \mathbb{D}$$

is, up to scaling, a bi-Lipschitz embedding with universal constants, so it is quasisymmetric with universal distortion. Since the composition of quasisymmetric maps is quasisymmetric, quantitatively, the result follows from Proposition 5. □

The following statement is an easy corollary of Proposition 7.

Corollary 8. *There is a universal constant* $0 < \beta_2 < 1/10$ *such that for any conformal embedding* $G\colon \mathbb{C}_1 \to \mathbb{D}$*,*

$$G(\mathbb{C}_{\beta_2}) \subseteq B_{\mathbb{D}}\left(G(0), \frac{l_G(0,1/2)}{6}\right).$$

Proof. By Proposition 7, there is a universal quasisymmetric distortion function η for $G\colon \mathbb{C}(1/10) \to \mathbb{D}$. Hence, if $\beta \in (0, 1/10)$, then

$$L_G(0,\beta) \leq \eta\left(10\beta\right) l_G(0,1/10) \leq \eta\left(10\beta\right) l_G(0,1/2).$$

Choosing β_2 so small that $\eta(10\beta_2) < 1/6$, we get the claim of the corollary. □

For the remainder of the paper, for convenience we define $\beta = \min\{\beta_1, \beta_2\}$.

2.4. **The data of X, scalings, and normalization.** Let X be a metric space that is Ahlfors 2-regular with constant K, linearly locally contractible with constant Λ, and homeomorphic to a compact orientable surface. We will refer to Λ and K as the *data* of X.

For $\lambda > 0$, we may form a new metric space X_λ by multiplying the original metric on X by λ. Then X_λ is again Ahlfors 2-regular and linearly locally contractible, and X_λ has the same data as X. Moreover, the identity mapping from X to X_λ is η-quasisymmetric with $\eta(t) = t$. Hence, in the proof of Theorem 2, we may scale the domain as we see fit.

For the remainder of this article, let X be a metric space that is Ahlfors 2-regular with constant K, linearly locally contractible with constant Λ, homeomorphic to a compact orientable surface, and has unit diameter. When we state that a quantity depends only the data of X, we are implicitly assuming this normalization.

The main reason for this convention (aside from notational convenience) is that Lemma 3 is not scale-invariant. Without this normalization, we would not be able to say that certain moduli of continuity depend only on the data of X.

3. Finding a conformal structure

By [Wil10, Theorem 4.1], X possesses a quasisymmetric atlas:

Theorem 9. *There is a quantity $A_0 \geq 1$ and a quasisymmetric distortion function η, each depending only on the data of X, such that for each $0 < R \leq 1/A_0$ there is a neighborhood U of x such that*

(1) *$B(x_0, R/A_0) \subseteq U \subseteq B(x_0, A_0 R)$,*
(2) *there exists an η-quasisymmetric map $\phi\colon U \to \mathbb{C}_1$ with $\phi(x_0) = 0$,*

The original theorem in [Wil10] does not include the normalization $\phi(x_0) = 0$. However, since $\phi\colon U \to \mathbb{C}_1$ is η-quasisymmetric, the basic distortion estimates [Hei01, Proposition 10.8] imply that $\phi(x_0) \in \mathbb{C}_\alpha$ where $\alpha < 1$ depends only on η. As the Möbius transformation $T(z) = \frac{z-\phi(x_0)}{1-\overline{\phi(x_0)}z}$ has quasisymmetric distortion depending only $|\phi(x_0)|$, we may assume that $\phi(x_0) = 0$. Accordingly, given a pair (U, ϕ) where $\phi\colon U \to \mathbb{C}_1$ is a homeomorphism, we will call $\phi^{-1}(0)$ the *center* of (U, ϕ).

We use the atlas provided by Theorem 9 to produce a conformal atlas on X that is adapted to its metric. We have separated the construction into two lemmas. The first is purely metric; the second modifies the output of the first.

Lemma 10. *Let $\rho \in (0,1)$ be given. Then there exists a quasisymmetric distortion function η, radii α, $r_0 > 0$, and a positive integer $n \in \mathbb{N}$, such that the following statements hold:*

(1) *There exists an atlas $\mathcal{A}_\rho = \{(U_j, \mathbb{C}_1, \phi_j) \mid j = 1, \ldots, n\}$ of X, where each mapping $\phi_j\colon U_j \to \mathbb{C}_1$ is an η-quasisymmetric homeomorphism with center denoted by x_j.*
(2) *The collection $\{B(x_j, r_0)\}_{j=1}^n$ is pairwise disjoint.*
(3) *The collection $\{B(x_j, 2r_0)\}_{j=1}^n$ covers X.*
(4) *For each $j = 1, \ldots, n$, it holds that $B(x_j, 10r_0) \subseteq U_j$, and*
$$\mathbb{C}_\alpha \subseteq \phi_j(B(x_j, r_0)) \subseteq \phi_j(B(x_j, 10r_0)) \subseteq \mathbb{C}_\rho.$$

Moreover, η depends only on the data of X, while α, r_0, and n depend only on the data of X and ρ.

Note that the collection $\{B(x_j, 10r_0)\}_{j=1}^n$ forms an open cover of X for which $8r_0$ is a Lebesgue number; cf. Theorem 4. Moreover, Lemma 3 implies that for each $j = 1, \dots, n$, the restriction $\phi_j|_{B(x_j, 10r_0)}$ and its inverse have moduli of continuity that depend only on ρ and the data of X.

Proof. For each $x \in X$, let $\phi_x \colon U_x \to \mathbb{C}_1$ be the η-quasisymmetric mapping provided by Theorem 9 with $R = 1/A_0$, so that

$$B(x, 1/A_0^2) \subseteq U_x \subseteq B(x, 1).$$

Applying Lemma 3 to ϕ_x and its inverse, we see that there is a common modulus of continuity ω for all of the mappings $\{\phi_x, \phi_x^{-1}\}_{x \in X}$, depending only on the data of X.

Hence, there is a radius $0 < r_0 < 1/10A_0^2$ and a number $0 < \alpha < \rho$, each depending only on ρ and the data of X, such that for each $x \in X$,

$$\phi_x(B(x, 10r_0)) \subseteq \mathbb{C}_\rho, \text{ and } \phi_x(B(x, r_0)) \supseteq \mathbb{C}_\alpha. \tag{2}$$

Let $\{x_1, \dots, x_n\}$ be a maximal $2r_0$-separated set in X. Then the open balls $\{B(x_j, r_0)\}_{j=1}^n$ are pairwise disjoint, while the open balls $\{B(x_j, 2r_0)\}_{j=1}^n$ cover X. Since X is Ahlfors 2-regular and we have assumed that $\operatorname{diam} X = 1$, this implies that n is comparable to r_0^{-2} and therefore depends only on ρ and the data. Moreover, as $10r_0 < 1/(A_0)^2$, for each $j = 1, \dots, n$, it holds that $B(x_j, 10r_0) \subseteq U_{x_j}$. In particular $\{U_{x_j}\}_{j=1}^n$ is also a cover of X. This shows that $\{(U_{x_j}, \mathbb{C}_1, \phi_{x_j})\}_{j=1}^n$ is the desired atlas. □

We now adapt the atlas given in Lemma 10 so that the transition mappings are conformal. This step is similar to the proof that a quasiconformal structure on a surface has a compatible conformal structure; see [Kuu67] and [Can69].

We recall that $\beta \in (0, 1)$ is a universal constant given by Corollaries 6 and 8.

Lemma 11. *There exists a quasisymmetric distortion function η, radii α, $r_0 > 0$, and a positive integer $n \in \mathbb{N}$, all depending only on the data of X, such that the following statements hold:*

1. *There exists an atlas $\mathcal{B} = \{(U_j, \mathbb{C}_1, \psi_j)\}_{j=1}^n$ of X, where each mapping $\psi_j \colon U_j \to \mathbb{C}_1$ is an η-quasisymmetric homeomorphism with center denoted by x_j.*
2. *The collection $\{B(x_j, r_0)\}_{j=1}^n$ is pairwise disjoint.*
3. *The collection $\{B(x_j, 2r_0)\}_{j=1}^n$ covers X.*
4. *For each $j = 1, \dots, n$, it holds that $B(x_j, 10r_0) \subseteq U_j$, and*
$$\mathbb{C}_\alpha \subseteq \psi_j(B(x_j, r_0)) \subseteq \psi_j(B(x_j, 10r_0)) \subseteq \mathbb{C}_\beta.$$
5. *The transition maps $\psi_j \circ \psi_k^{-1}$ are conformal wherever defined.*

Proof. We begin by letting $0 < \rho < 1$ be a number which will be determined below and will depend only on the data of X. Consider the atlas $\mathcal{A}_\rho$ provided by Lemma 10. Let us say that this atlas is given by η'-quasisymmetric charts

$\{\phi_j\colon U_j \to \mathbb{C}_1\}_{j=1}^n$ where η' depends only on the data of X. The associated radii α', r_0' and the number of charts n depend only on the data of X and ρ.

Since X is connected, the charts can be relabeled to satisfy

$$U_{j+1} \cap (U_1 \cup \ldots \cup U_j) \neq \emptyset$$

for $j = 1, \ldots, n-1$.

Define $\psi_1\colon U_1 \to \mathbb{C}_1$ by setting $\psi_1 = \phi_1$. We will iteratively construct $\psi_j\colon U_j \to \mathbb{C}_1$ for $j = 2, \ldots, n$ as follows. For $k < j$, we assume that the already constructed map ψ_k is quasisymmetric on U_k, has center x_k, and that if $k, k' < j$, then the transition functions $\psi_k \circ \psi_{k'}^{-1}$ are conformal where defined.

In the following, we will write $D_{a,b} := \phi_a(U_a \cap U_b)$ for indices $a > b$ in $\{1, \ldots, n\}$.

For $k < j$, define $T_{j,k} = \psi_k \circ \phi_j^{-1}$. Then $T_{j,k}$ is quasisymmetric and hence quasiconformal on $D_{j,k}$. Therefore, the complex dilatation $\mu_{j,k}$ of $T_{j,k}$ is well-defined (up to a.e. equivalence) on $D_{j,k}$. Given another index $k' < j$, it holds that $T_{j,k} = (\psi_k \circ \psi_{k'}^{-1}) \circ T_{j,k'}$ on $D_{j,k} \cap D_{j,k'}$. It therefore follows from the conformality of $\psi_k \circ \psi_{k'}^{-1}$ that $\mu_{j,k} = \mu_{j,k'}$ a.e. on the intersection $D_{j,k} \cap D_{j,k'}$. This shows that there exists a measurable Beltrami coefficient $\mu_j\colon \mathbb{C}_1 \to \mathbb{C}_1$ with $\mu_j = \mu_{j,k}$ a. e. on $D_{j,k}$ for each index $1 \leq k < j$, and $\mu_j = 0$ on $\mathbb{C}_1 \setminus \bigcup_{k=1}^{j-1} D_{j,k}$. We extend μ_j to the whole plane by $\mu_j(z) = (z/\overline{z})^2\, \overline{\mu_j(1/\overline{z})}$ for $|z| > 1$. By the Measurable Riemann Mapping Theorem there exists a unique quasiconformal map $h_j\colon \mathbb{C} \to \mathbb{C}$, normalized by $h_j(0) = 0$, $h_j(1) = 1$, with complex dilatation $\mu_{h_j} = \mu_j$ a.e. By the symmetry of μ_j and the chosen normalization, $h_j(1/\overline{z}) = 1/\overline{h_j(z)}$, and so $h_j(\mathbb{C}_1) = \mathbb{C}_1$. We define $\psi_j = h_j \circ \phi_j$. The transformation formula for Beltrami coefficients (see, e.g., [LV73, IV.5.1]) shows that if $k < j$, then $\psi_k \circ \psi_j^{-1} = T_{j,k} \circ h_j^{-1}$ is conformal. Moreover, $\psi_j(x_j) = 0$.

We claim that for each $j = 1, \ldots, n$, the mapping ψ_j has a quasisymmetric distortion function that depends only on the data of X; as this is true of ϕ_j, it suffices to prove the same of h_j, and we may also assume that $j > 1$. As a normalized quasiconformal self-map of the unit disk, h_j is quasisymmetric with distortion controlled by the maximal dilatation $\|\mu_j\|_\infty$, see e. g. [Väi81, Theorem 2.4]. By an inductive argument, it is easy to show that this dilatation is bounded by a constant depending only on η' (which depends only on the data of X) and the number of charts n (which depends on the data and ρ). However, the bound is actually independent of ρ, by the following argument.

Fix $j > 1$. By the uniform quasisymmetry of $\{\phi_k\}_{k=1}^n$, there is a quantity $0 \leq \kappa < 1$, depending only on the data of X, such that for all indices $k < j$

with $D_{j,k} \neq \emptyset$, the complex dilatation $\mu_{\phi_k \circ \phi_j^{-1}}$ of $\phi_k \circ \phi_j^{-1}\colon D_{j,k} \to \phi_k(D_{j,k})$ satisfies $\|\mu_{\phi_k \circ \phi_j^{-1}}\|_\infty \leq \kappa$. For $k \in \{1, \ldots, j-1\}$, define

$$F_{j,k} = D_{j,k} \setminus \bigcup_{l=1}^{k-1} D_{j,l}, \ F_k = \mathbb{C}_1 \setminus \bigcup_{l=1}^{k-1} D_{k,l}, \text{ and } F_j = \mathbb{C}_1 \setminus \bigcup_{k=1}^{j-1} D_{j,k}.$$

If $z \in F_j$, then $\mu_j(z) = 0$. If $z \in F_{j,k}$, then $\phi_k \circ \phi_j^{-1}(z) \in F_k$, and hence

$$\mu_k|_{\phi_k \circ \phi_j^{-1}(F_{j,k})} = 0.$$

The transformation formula for Beltrami coefficients now shows that for almost-every point $z \in F_{j,k}$,

$$\mu_j(z) = \mu_{j,k}(z) = \mu_{\psi_k \circ \phi_j^{-1}}(z) = \mu_{h_k \circ \phi_k \circ \phi_j^{-1}}(z) = \mu_{\phi_k \circ \phi_j^{-1}}(z).$$

Since $\mathbb{C}_1 = \bigcup_{k=1}^{j-1} F_{j,k} \cup F_j$, we see that $\|\mu_j\|_\infty \leq \kappa$. As discussed above, we may now conclude that each of the mappings $\{\psi_j\}_{j=1}^n$ has a quasisymmetric distortion function that depends only on the data of X.

We have now seen the atlas $\mathcal{B} := \{(U_j, \mathbb{C}_1, \psi_j) | j = 1, \ldots, n\}$ of X satisfies conditions (1) and (5) of the statement. Moreover, setting $r_0 := r_0'$, the conditions (2) and (3) follow directly from the corresponding statements for the atlas $\mathcal{A}_{X,\rho}$. Note that only condition (4) involves the constant β. We now show how to choose ρ so that condition (4) is satisfied.

Let us make the convention that $h_1\colon \mathbb{C}_1 \to \mathbb{C}_1$ is the identity. As discussed above, the mappings $\{h_j\}_{j=1}^n$ are uniformly quasisymmetric with a distortion function depending only on the data of X. Hence, by Lemma 3, there is a common modulus of continuity for all of the mappings $\{h_j, h_j^{-1}\}_{j=1}^n$ that depends only on the data of X. Since β is a universal constant, we may choose $\rho > 0$ depending only on the data of X such that for each $j = 1, \ldots, n$, it holds that $h_j(\mathbb{C}_\rho) \subseteq \mathbb{C}_\beta$. Having so chosen ρ, the radius α' depends only on the data of X, and so we may also choose $\alpha > 0$ depending only on the data of X such that for for each $j = 1, \ldots, n$, $h_j(\mathbb{C}_{\alpha'}) \supseteq \mathbb{C}_\alpha$. This establishes condition (4). □

3.1. **Uniformizing to a standard metric.** The atlas $\mathcal{B}$ given by Lemma 11 determines a conformal structure on the compact orientable surface X, i.e., the pair $(X, \mathcal{B})$ determines a Riemann surface. By the classical uniformization theorem, $(X, \mathcal{B})$ is conformally equivalent to a standard Riemann surface $Y = U/\Gamma$, where U denotes the standard Riemann surface structure on the sphere, the plane, or the unit disk, and Γ is a discrete group of Möbius transformations acting freely and properly discontinuously on U. The standard spherical, plane, or hyperbolic Riemannian metric on U then descends to a Riemannian metric of constant curvature $+1$, 0, or -1 on Y, compatible with the conformal structure. We fix a uniformizing conformal homeomorphism $F\colon (X, \mathcal{B}) \to Y$, and equip Y with the distance function d_Y arising from the Riemannian metric of constant curvature. Denote by π the quotient mapping from U to Y. Recall that d_Y may be expressed as $d_Y(p,q) = \inf_\gamma \operatorname{length}(\gamma)$, where the infimum is taken over all smooth paths γ in U such that the projected path $\pi \circ \gamma$ connects p and q. A priori, it is not clear how the properties of the distance d_Y or the map F depend on the original metric space (X, d). The following statement is the main result of this paper, and completes the proof of Theorem 2.

Theorem 12. *The uniformizing map $F\colon (X, d) \to (Y, d_Y)$ is η-quasisymmetric, with distortion η depending only on the data of X.*

Proof of Theorem 12. In the case that $U = \hat{\mathbb{C}}$, then X itself must be homeomorphic to $\hat{\mathbb{C}}$, and so Theorem 1 implies Theorem 12. We consider the remaining planar and hyperbolic cases together.

Define $g_j\colon \mathbb{C}_1 \to F(U_j)$ by $g_j = F \circ \psi_j^{-1}$. Then g_j is a conformal homeomorphism that lifts to a conformal embedding $G_j\colon \mathbb{C}_1 \to U$.

We use a sequence of claims to complete the proof.

Claim 1. Let $j \in \{1, \dots, n\}$. The projection $\pi\colon U \to Y$ maps $G_j(\mathbb{C}_\beta)$ isometrically onto $g_j(\mathbb{C}_\beta)$. In other words, for each $z, z' \in \mathbb{C}_\beta$, $|g_i(z) - g_i(z')|_Y = |G_i(z) - G_i(z')|_U$.

Proof of Claim 1. For $u \in U$, define

$$r(u) = \min\{|u - \gamma(u)|_U : \gamma \in \Gamma \setminus \{\mathrm{id}\}\}.$$

If $U = \mathbb{C}$, then $r(u)$ is independent of $u \in U$ and depends only on the group Γ. If $U = \mathbb{D}$, then $r(u)$ is bounded below by the minimal translation distance of a non-identity element of $\Gamma \setminus \{\mathrm{id}\}$, and is a 2-Lipschitz function of z, see e. g. [Bea83, Section 7.35]. These facts imply that for any $u \in U$ and $r > 0$,

- if $r \leq r(u)/6$, then $\pi|_{B_U(u,r)}$ is an isometry,
- if $\pi|_{B_U(u,r)}$ is injective, then $r \leq r(u)$.

Since $\pi\colon G_j(\mathbb{C}_1) \to g_j(\mathbb{C}_1)$ is injective, it follows that $l_{G_j}(0,1) \leq r(G_j(0))$. Corollaries 6 and 8 now complete the proof of this claim. □

Claim 2. For each $j = 1, \dots, n$, the mapping $F|_{B(x_j, 10r_0)}$ is quasisymmetric with distortion that depends only on the data of X.

Proof of Claim 2. By Propositions 5 and 7, the mapping G_j restricted to $\mathbb{C}_\beta$ is quasisymmetric with a universal distortion function. By Claim 1, this is also true of g_j. Since we may write $F = g_j \circ \psi_j$, and ψ_j is η-quasisymmetric where η depends only on the data, it follows that for each $j = 1, \dots, n$, the mapping $F|_{\psi_j^{-1}(\mathbb{C}_\beta)}$ is quasisymmetric with distortion that depends only on the data of X. According to Lemma 11, $\psi_j^{-1}(\mathbb{C}_\beta) \supseteq B(x_j, 10r_0)$, implying the claim. □

Claim 3. There is a constant $C \geq 1$ depending only on the data of X such that for each $j = 1, \ldots, n$,

$$l_{g_j}(0, \alpha) \geq \frac{\operatorname{diam} Y}{C}. \tag{3}$$

Proof of Claim 3. As $\{B_X(x_j, 2r_0)\}_{j=1}^n$ covers X and $\psi_j(B_X(x_j, 10r_0)) \subseteq \mathbb{C}_\beta$, it holds that $\{\psi_j^{-1}(\mathbb{C}_\beta)\}_{j=1}^n$ also covers X. Since F is a homeomorphism, this implies that $Y = \bigcup_{j=1}^n g_j(\mathbb{C}_\beta)$. As Y is connected, it follows that

$$\max\{\operatorname{diam} g_j(\mathbb{C}_\beta) : j = 1, \ldots, n\} \geq \frac{\operatorname{diam} Y}{n}. \tag{4}$$

Recall that the number of charts n depends only on the data of X.

Consider indices j and k in $\{1, \ldots, n\}$, and suppose that $|x_j - x_k|_X < 4r_0$. Then $\psi_j(x_k) \in \mathbb{C}_\beta$, and so $d_Y(F(x_j), F(x_k)) \leq L_{g_j}(0, \beta)$. On the other hand, $|x_j - x_k| \geq r_0$, and so $F(x_k) \notin g_j(\mathbb{C}_\alpha)$, implying $d_Y(F(x_j), F(x_k)) \geq l_{g_j}(0, \alpha)$. Moreover, since $g_j \colon \mathbb{C}_\beta \to Y$ is a quasisymmetric embedding with universal distortion function, there is a quantity $C_0 \geq 1$ depending only on the ratio of β to α, and hence only on the data of X, such that

$$l_{g_j}(0, \alpha) \geq \frac{L_{g_j}(0, \beta)}{C_0} \geq \frac{\operatorname{diam} g_j(\mathbb{C}_\beta)}{2C_0}$$

Since in addition,

$$\operatorname{diam} g_j(\mathbb{C}_\beta) \geq L_{g_j}(0, \beta) \geq l_{g_j}(0, \alpha),$$

we have now shown that the quantities

$$l_{g_j}(0, \alpha), \; L_{g_j}(0, \beta), \; \operatorname{diam} g_j(\mathbb{C}_\beta), \text{ and } d_Y(F(x_j), F(x_k))$$

are all comparable with constants that depend only on the data of X. The same is true with the roles of j and k reversed.

Since the open sets $\{B(x_j, 2r_0)\}_{j=1}^n$ cover the connected space X, for any pair of indices $j, j' \in \{1, \ldots, n\}$, there is a sequence of distinct indices $j = j_1, \ldots, j_k = j'$ of length at most n so that $|x_{j_i} - x_{j_{i+1}}| < 4r_0$ for each $i = 1, \ldots, k-1$. Since n depends only on the data of X, (4) proves the claim. □

We complete the proof of Theorem 12 by employing Theorem 4. We consider the covering of X given by $\{B(x_j, 10r_0) : j = 1, \ldots, n\}$. By Claim 2, the mapping F is quasisymmetric on each element of this cover with a distortion function η that depends only on the data of X. Moreover, Lemma 11 implies that this cover has Lebesgue number $8r_0$. Suppose that $x, x' \in X$ satisfy $|x - x'| = 4r_0$. We may find indices j and k in $\{1, \ldots, n\}$ such that $|x - x_j| < 2r_0$ and $|x' - x_k| < 2r_0$. Then $2r_0 \leq |x' - x_j| \leq 6r_0$, so $x, x', x_j \in B(x_j, 10r_0)$ and

$$|F(x') - F(x)| \geq \frac{|F(x') - F(x_j)|}{\eta\left(\frac{|x' - x_j|}{|x' - x|}\right)} \geq \frac{|F(x') - F(x_j)|}{\eta\left(\frac{3}{2}\right)}$$

Since $x' \notin B(x_j, r_0)$, we see that $F(x') \notin g_j(\mathbb{C}_\alpha)$, so by Claim 3,

$$|F(x') - F(x_j)| = |F(x') - g_j(0)| \geq l_{g_j}(0, \alpha) \geq \frac{\operatorname{diam} Y}{C}.$$

Now Theorem 4 implies that $F\colon X \to Y$ is quasisymmetric with distortion function depending only on the data of X. □

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