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QUASISYMMETRIC STRUCTURES ON SURFACES

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In memoriam: Juha Heinonen (1960 - 2007)

Abstract. We show that a locally Ahlfors 2-regular and locally linearly locally contractible metric surface is locally quasisymmetrically equivalent to the disk. We also discuss an application of this result to the problem of characterizing surfaces embedded in some Euclidean space that are locally bi-Lipschitz equivalent to a ball in the plane.

1. Introduction

Quasisymmetric mappings are a generalization of conformal mappings to metric spaces. Recent work [4], [32], [23] has provided a substantial existence theory, analogous to the classical Uniformization Theorem for Riemann surfaces, for quasisymmetric mappings on metric spaces that satisfy simple geometric conditions. Quasisymmetric uniformization results have been applied to diverse subjects, including Cannon's conjecture regarding Gromov hyperbolic groups with two-sphere boundary [4] and the quasiconformal Jacobian problem [3]. In addition, the study of existence and ubiquity of quasisymmetric mappings has been extended to Sierpinski carpets [7], [6] .

In this paper, we develop a local existence theory for quasisymmetric mappings on general surfaces. In other words, we give simple geometric conditions for a metric space homeomorphic to a surface to be considered a generalized Riemann surface. The geometric conditions we consider are localized versions of Ahlfors regularity and linear local contractibility. We defer the precise definitions to Section 3.

The global versions of these conditions have been studied extensively [27], [4], [32]. A deep theorem of Semmes [27, Theorem B.10] states that if a metric space homeomorphic to a connected, orientable, n-manifold is complete, Ahlfors n-regular, and linearly locally contractible, then it supports a weak $(1,1)$-Poincaré inequality. Spaces that support such an inequality can be considered to have "good calculus", and they are the preferred environment for the theory of quasiconformal mappings on metric spaces [15]. Moreover, they enjoy several geometric properties, such as quasiconvexity [19].

Our main result is the following theorem.

Theorem 1.1. *Let (X,d) be a locally Ahlfors 2-regular and locally linearly locally contractible (LLLC) metric space homeomorphic to a surface. Then each point of X has a neighborhood that is quasisymmetrically equivalent to the disk.*

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Our proof is in fact quantitative, and provides good bounds on the size of the resulting quasidisk. See Theorem 4.1 for the complete result. An outline of the proof of Theorem 1.1 is as follows. The main task is to construct an Ahlfors 2-regular and linearly locally connected (a slightly weaker condition than linear local contractibility) planar neighborhood of a given point $z \in X$. We then apply previously established uniformization results from [32] to produce the desired quasisymmetric mapping. The obstacle is that compact subsets of a locally Ahlfors 2-regular and $LLLC$ metric space need not be Ahlfors 2-regular and linearly locally connected. However, we show that if γ is a quasicircle contained in a planar subset of X, then the closed Jordan domain defined by γ is Ahlfors 2-regular and linearly locally connected. Thus it suffices to construct a quasicircle at a specificed scale that surrounds a given point $p \in X$.

In constructing the quasicircle, we first show that (X, d) is locally quasiconvex. This would follow from the result of Semmes [27, Theorem B.10], except that we consider localized conditions. As it is, our methods resemble those employed by Semmes. As we have specialized to two dimensions, our proof is fairly elementary and direct. We also indicate how our proof could be upgraded to give a full local analogue of Semmes' result in two dimensions.

A locally compact and locally quasiconvex space is, up to a locally bi-Lipschitz change of metric, locally geodesic. With this simplification, we employ discrete methods to construct a loop surrounding z of controlled length in a controlled annulus. We then solve an extremal problem to produce the desired quasicircle.

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This paper is dedicated to Juha Heinonen, who passed away shortly after it was submitted. Juha was a great thinker, a fantastic mentor, and an excellent friend. He is sorely missed.

2. Application to local bi-Lipschitz parameterizations

Theorem 1.1 plays a role in the program of Heinonen and others to give necessary conditions for an n-dimensional submanifold of some $\mathbb{R}^N$ to be locally bi-Lipschitz equivalent to a ball in $\mathbb{R}^n$. We give a brief description of this program. See [14], [16], and [17] for a full exposition.

An oriented, n-dimensional submanifold X of $\mathbb{R}^N$ admits local Cartan-Whitney presentations if for each point $p \in X$ there is an n-tuple of flat 1-forms $\rho = (\rho_1, \ldots, \rho_n)$ defined on an $\mathbb{R}^N$ neighborhood of p, such that near p on X, there is a constant $c > 0$ such that

$$*(\rho_1 \wedge \ldots \wedge \rho_n) \geq c > 0.$$

Here $*$ denotes the Hodge star operator.

Theorem 2.1 (Heinonen-Sullivan [17])**.** *Let $X \subseteq \mathbb{R}^N$ be an n-manifold endowed with the metric inherited from $\mathbb{R}^N$. If X is locally Ahlfors n-regular, $LLLC$, and admits local Cartan-Whitney presentations, then X is locally bi-Lipschitz equivalent to a ball in $\mathbb{R}^n$, except on a closed set of measure* 0 *and topological dimension at most $n - 2$.*

The basic outline of the proof of Theorem 2.1 is as follows. Given $p \in X$, let ρ be a Cartan-Whitney presentation near $p \in X$. It is shown that on a sufficiently small open neighborhood U of p, the map $f\colon U \to \mathbb{R}^n$ defined by

$$f(x) = \int_{[p,x]} \rho \tag{2.1}$$

is of bounded length distortion. This implies that it is quasiregular (and so continuous, discrete, and open), and that the volume derivative of f uniformly bounded away from 0. It follows that f is a branched covering that is locally bi-Lipschitz off its branch set, which is of measure 0 and topological dimension at most $n-2$.

If $n = 2$, then this branch set consist of isolated points. Bonk and Heinonen [14] noted that in this case, the measurable Riemann mapping theorem and and a statement like Theorem 1.1 would provide a resolution of this branch set.

Theorem 2.2 (Bonk-Heinonen)**.** *Let $X \subseteq \mathbb{R}^N$ be a surface endowed with the metric inherited from $\mathbb{R}^N$. If X is locally Ahlfors 2-regular, LLLC, and admits local Cartan-Whitney presentations, then X is locally bi-Lipschitz equivalent to a ball in $\mathbb{R}^2$.*

As no proof of Theorem 2.2 is available in the literature, we now sketch a proof of Theorem 2.2 based on conversations with Bonk and Heinonen.

We begin with a topological lemma regarding continuous, discrete, and open maps.

Lemma 2.3. *Let U be a topological space homeomorphic to $\mathbb{R}^2$, and let $p \in U$. Let $f\colon U \to \mathbb{R}^2$ be a continuous, discrete, and open map with $f(p) = 0$. For $\epsilon > 0$, denote the p-component of $f^{-1}(B_{\mathbb{R}^2}(0,\epsilon))$ by V_ϵ. Then there is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, the set V_ϵ is simply connected.*

Proof. Elementary properties of continuous, discrete, and open mappings [26, Lemma I.4.9] show that there is an $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$, the set V_ϵ has compact closure, the image of V_ϵ is $B_{\mathbb{R}^2}(0,\epsilon)$, and the image of the topological boundary of V_ϵ is $\partial B_{\mathbb{R}^2}(0,\epsilon)$.

Let $0 < \epsilon < \epsilon_0$, and suppose that V_ϵ fails to be simply connected. Then it has at least one compact complementary component. It follows from the openness of f that if W is such a compact complementary component, then the image of the topological boundary of W is a non-empty subset of $\partial B_{\mathbb{R}^2}(0,\epsilon)$. Since f is continuous, the set $f(W)$ is compact and hence $f(W) \subseteq \bar{B}(0,\epsilon)$. Let V' be the union of V_ϵ and all of its compact complementary components. Then $\mathbb{R}^2 \backslash V'$ is the unique non-compact component of $\mathbb{R}^2 \backslash V_\epsilon$, and hence it is a closed set. Thus V' is an open set, but $f(V')$ is not. This is a contradiction with the openess of f. □

Proof of Theorem 2.2. Let $p \in X$. We may find an open neighborhood U of p on which the map f given by (2.1) is well defined. Denote the local degree of f at p by the integer $k \geq 1$. By definition, $f(p) = 0 \in \mathbb{R}^2$.

By Theorem 1.1, there is an open neighborhood $U' \subseteq U$ of p and a quasisymmetric homeomorphism $\phi\colon U' \to \mathbb{D}^2$. We may assume without loss of generality that $\phi(p) = 0$. By Lemma 2.3, there is an $\epsilon > 0$ so that $V_\epsilon \subseteq U'$ is simply connected. Set $\Omega = \phi(V_\epsilon)$. Then $h := f \circ \phi^{-1}$ is a quasiregular mapping from the simply connected domain $\Omega \subseteq \mathbb{D}^2$ to $B_{\mathbb{R}^2}(0,\epsilon)$. Thus h defines a Beltrami differential μ_h on Ω. By the measurable Riemann mapping theorem, there is a quasiconformal mapping $g\colon \Omega \to B_{\mathbb{R}^2}(0,\epsilon)$ such that the Beltrami differential μ_g agrees with μ_h almost everywhere. The uniqueness statement of the measurable Riemann mapping theorem implies that $h(z) = g(z)^k$. Let $\rho\colon B_{\mathbb{R}^2}(0,\epsilon) \to B_{\mathbb{R}^2}(0,\epsilon^k)$ be the radial stretching map

$$\rho(z) = |z|^{k-1}z,$$

and define a homeomorphism $\psi\colon U' \to B_{\mathbb{R}^2}(0,\epsilon^k)$ by $\psi = \rho \circ g \circ \phi$. Then we have the following relationship amongst Jacobian determinants:

$$J_f = J_h J_\phi = kJ_{\rho\circ g}J_\phi = kJ_\psi.$$

Hence the volume derivative of ψ is bounded above and below by a constant multiple of the volume derivative of f. Since ψ is quasiconformal, this implies that ψ is bi-Lipschitz, completing the proof. □

3. Notations, definitions and preliminary results

3.1. **Metric spaces.** We will often denote a metric space (X,d) by X. Given a point $x \in X$ and a number $r > 0$, we define the open and closed balls centered at x of radius r by

$$B_{(X,d)}(x,r) = \{y \in X : d(x,y) < r\} \quad \text{and} \quad \bar{B}_{(X,d)}(x,r) = \{y \in X : d(x,y) \leq r\}.$$

Where it will not cause confusion, we denote $B_{(X,d)}(x,r)$ by $B_X(x,r)$, $B_d(x,r)$, or $B(x,r)$. A similar convention is used for all other notions which depend implicitly on the underlying metric space. The diameter of a subset E of (X,d) is denoted by $\operatorname{diam}(E)$, and the distance between two subsets $E,F \subseteq X$ is denoted by $\operatorname{dist}(E,F)$. For $\epsilon > 0$, the ϵ-neighborhood of $E \subseteq X$ is given by

$$\mathcal{N}_\epsilon(E) = \bigcup_{x\in E} B(x,\epsilon).$$

We denote the completion of a metric space X by $\bar{X}$, and define the metric boundary of X by $\partial X = \bar{X}\backslash X$. These notions are not to be confused with their topological counterparts. Given a subset E of a topological space X, the closure $\operatorname{cl}_X(E)$ of E in X is defined to be the intersection of all closed subsets of X containing E. The frontier or topological boundary of E in X is defined by

$$\operatorname{fr}_X(E) = \operatorname{cl}_X(E) \cap \operatorname{cl}_X(X\backslash E).$$

Note that the frontier of a closed set need not be empty, while the metric boundary of a complete set is always empty.

Let $[a,b] \subset \mathbb{R}$ be a compact interval. A continuous map $\gamma\colon [a,b] \to X$ is called a path in X. The path γ may also be referred to as a parameterization of its image $\operatorname{im}\gamma$. If γ happens to be an embedding, then $\operatorname{im}\gamma$ is called an arc in X. If α is an arc in X, and $u,v \in \alpha$, then the segment $\alpha[u,v] \subseteq X$ is well defined.

A path $\gamma\colon [a,b] \to X$ is rectifiable if $\operatorname{length}(\gamma) < \infty$. The metric space (X,d) is L-quasiconvex, $L \geq 1$, if every pair of points $x,y \in X$ may be joined by a path in X of length no more than $Ld(x,y)$. We say that (X,d) is locally quasiconvex if for each point $z \in X$ there is a constant L_z and a neighborhood U of z such that each pair of points $x,y \in U$ maybe connected by a path in X of length no more than $L_z d(x,y)$.

Given a rectifiable path γ of length L, and a Borel function $\rho\colon X \to [0,\infty]$, we define the path integral

$$\int_\gamma \rho \; ds = \int_0^L (\rho \circ \gamma_l)(t) \; dt,$$

where γ_l denotes the arclength reparameterization of γ.

If γ is a path connecting points $x, y \in X$ and satisfying

$$d(x,y) = \text{length}(\gamma),$$

then γ is said to be a geodesic. If the points $x, y \in X$ may be connected by a geodesic, we define the geodesic segment $[x,y]$ to be the image of some chosen geodesic with endpoints x and y. By convention, we assume that $[x,y] = [y,x]$ as sets, and that if $w \in [x,y]$, then $[x,w]$ and $[w,y]$ are subsets of $[x,y]$. Given a geodesic segment $[x,y]$, we denote by $s^{[x,y]} \colon [0, d(x,y)] \to X$ the arc length parameterization of $[x,y]$, with the convention that $s^{[x,y]}$ has initial value x and terminal value y. If $[x,y]$ is a geodesic segment with $x \neq y$ and $[a,b] \subseteq \mathbb{R}$ is a compact non-degenerate interval, we define the standard parameterization of $[x,y]$ by $[a,b]$ to be given by $s^{[x,y]}_{[a,b]} \colon [a,b] \to X$ where

$$s^{[x,y]}_{[a,b]}(t) = s^{[x,y]}\left(\frac{t-a}{b-a} d(x,y)\right).$$

Metric spaces admit arbitrarily fine approximations by discrete spaces in the following sense. Given $\epsilon > 0$, a subset $S \subseteq X$ is ϵ-separated if $d(a,b) \geq \epsilon$ for all pairs of distinct points $a, b \in S$. By Zorn's lemma, maximal ϵ-separated sets exist for every $\epsilon > 0$, and for such sets the collection $\{B(a,\epsilon)\}_{a \in S}$ covers X.

We denote by $\mathbb{S}^2, \mathbb{R}^2$, and $\mathbb{D}^2$ the sphere, the plane, and the disk, each equipped with the standard metric inherited from the ambient Euclidean space. In this paper, a surface is a connected two-dimensional topological manifold.

For specificity, we define $\mathbb{S}^1 := [0, 2\pi)$ as a set, and topologized and metrized as a subset of the plane under the identification $\theta \leftrightarrow e^{i\theta}$. A continuous map of $\mathbb{S}^1$ to a space X is called a loop in X. We define length and integrals for loops as for paths, with the obvious modifications. A collection of points $\{\theta_1, \dots, \theta_n\} \subseteq \mathbb{S}^1$ is said to be in cyclic order if they are ordered according to the standard positive orientation on $\mathbb{S}^1$. Given a cyclically ordered collection of points $\{\theta_1, \dots, \theta_n\}$ containing at least three distinct points, we may unambiguously define the arcs $[\theta_i, \theta_{i+1}]$, $i = 1, \dots, n, \bmod n$, that lie between consecutive points.

3.2. **Finite dimensional metric spaces.** A metric space (X,d) is said to be doubling if there exists a non-negative integer N such that for each $a \in X$ and $r > 0$, the ball $B(a,r)$ may be covered by at most N balls of radius $r/2$. If (X,d) is doubling, then we may find constants $Q \geq 0$ and $C \geq 1$, depending only on N, such that for all $0 < \epsilon \leq 1/2$, each ball $B(a,r)$ may be covered by at most $C\epsilon^{-Q}$ balls of radius ϵr. The infimum over such Q is called the Assouad dimension of (X,d).

For $Q \geq 0$, we will denote the Q-dimensional Hausdorff measure on (X,d) by $\mathcal{H}^Q_{(X,d)}$. For a full description of Hausdorff measure, see [11, 2.10].

Definition 3.1. A metric space (X,d) is called Ahlfors Q-regular, $Q \geq 0$, if there exists a constant $C \geq 1$ such that for all $a \in X$ and $0 < r \leq \operatorname{diam} X$, we have

$$\frac{r^Q}{C} \leq \mathcal{H}^Q(\bar{B}_d(a,r)) \leq Cr^Q. \tag{3.1}$$

Note that if the upper bound in (3.1) is valid for all $0 < r \leq \operatorname{diam}(X)$, then it is also valid for all $r > \operatorname{diam}(X)$ as well.

An Ahlfors Q-regular metric space can be thought of as Q-dimensional at every scale. For example, the space $\mathbb{R}^2$ is Ahlfors 2-regular, while the infinite strip

$$\{(x,y)\in\mathbb{R}^2 : 0<y<1\}$$

is not Ahlfors Q-regular for any Q. At small scales, the strip appears two-dimensional, while at large scales it appears one-dimensional. For an in-depth discussion of Ahlfors regularity, see [27, Appendix C].

Definition 3.2. A metric space (X,d) is locally Ahlfors Q-regular, $Q\geq 0$, if for every compact set $K\subseteq X$ there exists a constant $C_K\geq 1$ and a radius $R_K>0$ such that for all $x\in K$ and $0<r\leq R_K$, we have

$$C_K{}^{-1}r^Q \leq \mathcal{H}^Q(\bar{B}_d(x,r)) \leq C_K r^Q. \tag{3.2}$$

Note that this definition is localized in two ways: the constant C_K depends on the location of the center of the ball under consideration, and the radius R_K restricts the scales to which the condition applies at this location. We will only apply this definition to spaces which have many compact subsets, i.e., locally compact spaces.

It will be convenient to have a notion where the radius R_K is tied to the size of the set under consideration.

Definition 3.3. A subset U of a metric space (X,d) is relatively Ahlfors Q-regular, if there exists a constant $C\geq 1$ such that for all $0<r\leq \operatorname{diam}(U)$ and all $x\in U$,

$$C^{-1}r^Q \leq \mathcal{H}^Q(\bar{B}_X(x,r)) \leq Cr^Q. \tag{3.3}$$

Note that in the definition of a relatively Ahlfors regular set U, the balls under consideration may contain points outside of U; we require (3.3) to hold for $B_X(x,r)$ and not $B_U(x,r)$. Hence a relatively Ahlfors regular set need not be Ahlfors regular as a metric space.

To state some of our theorems in full generality, we also employ a relative doubling condition.

Definition 3.4. The relative Assouad dimension of a subset U of a metric space (X,d) is the infimum of all $Q\geq 0$ such that there exists a constant $D\geq 1$ with the property that for all $0<r\leq \operatorname{diam}(U)$, all $x\in U$, and all $0<\epsilon\leq 1/2$, the ball $B_X(x,r)$ can be covered by at most $D\epsilon^{-Q}$ balls in X of radius ϵr.

Remark 3.5. The infimum in the definition of relative Assouad dimension need not be attained. If there are constants $Q\geq 0$ and $D\geq 1$ with the property that for all $0<r\leq \operatorname{diam}(U)$, all $x\in U$, and all $0<\epsilon\leq 1/2$, the ball $B_X(x,r)$ can be covered by at most $D\epsilon^{-Q}$ balls in X of radius ϵr, then we say that U has relative Assouad dimension at most Q with constant D. If this Q happens to be the relative Assouad dimension of U, then we say that U has relative Assouad dimension Q achieved with constant D.

We now give a local version of the fact that Ahlfors regular spaces are doubling.

Proposition 3.6. *Let (X,d) be a locally Ahlfors Q-regular metric space, and let $K\subseteq X$ be compact. Let R_K and C_K be the constants associated to K by the local Ahlfors Q-regularity condition. If U is a subset of X such that the $2\operatorname{diam}(U)$-neighborhood of U is contained in K and $\operatorname{diam}(U)\leq R_K/2$, then U has relative Assouad dimension Q achieved with constant $C_K^2 8^Q$.*

Proof. Let $x \in U$, $0 < r \leq \operatorname{diam} U$, and $0 < \epsilon \leq 1/2$. Then $x \in K$, and $0 < r \leq R_K/2$. Let $\{x_i\}_{i\in I}$ be a maximal ϵr-separated set in $B(x,r)$. Then $\{B(x_i,\epsilon r)\}_{i\in I}$ covers $B(x,r)$, while $\{\bar{B}(x_i,\epsilon r/4)\}_{i\in I}$ is disjointed. Since $\epsilon < 1/2$, we see that for all $i \in I$, $B(x_i,\epsilon r) \subseteq \bar{B}(x,2r)$. Since the $2\operatorname{diam}(U)$-neighborhood of U is contained in K, we see that $x_i \in K$. We may now apply the local Ahlfors Q-regularity condition to see that

$$\frac{\operatorname{card}(I)}{C_K}\left(\frac{\epsilon r}{4}\right)^Q \leq \sum_{i\in I} \mathcal{H}^Q\left(\bar{B}(x_i,\epsilon r/4)\right) \leq \mathcal{H}^Q(\bar{B}(x,2r)) \leq C_K(2r)^Q.$$

This implies that

$$\operatorname{card}(I) \leq C_K^2 8^Q \epsilon^{-Q},$$

showing that the relative Assouad dimension of U is at most Q and giving the desired constant. The fact that the relative Assouad dimension of U is precisely Q is similarly straight-forward. Since it will not actually be needed later, we leave the proof to the reader. □

3.3. **Contractibility and connectivity conditions.** Here we discuss various types of quantitative local connectivity and contractibility. Perhaps the most basic is the following.

Definition 3.7. A subset E of a metric space (X,d) is of λ-bounded turning in X, $\lambda \geq 1$, if each pair of distinct points $x, y \in E$ may be connected by a continuum $\gamma \subseteq X$ such that $\operatorname{diam}(\gamma) \leq \lambda d(x,y)$. If E is bounded turning in itself, then it is said to be of bounded turning.

Recall that a continuum is a compact connected set containing at least two points. The condition for a subset E to be of bounded turning in a metric space (X,d) is non-standard; usually only spaces which are of bounded turning in themselves are considered. The bounded turning condition along with a similar dual condition constitute linear local connectivity.

Definition 3.8. Let $\lambda \geq 1$. A metric space (X,d) is λ-linearly locally connected (λ-*LLC*) if for all $a \in X$ and $r > 0$, the following two conditions are satisfied:

(i) for each pair of distinct points $x, y \in B(a,r)$, there is a continuum $E \subseteq B(a,\lambda r)$ such that $x, y \in E$,
(ii) for each pair of distinct points $x, y \in X \backslash B(a,r)$, there is a continuum $E \subseteq X \backslash B(a,r/\lambda)$ such that $x, y \in E$.

Individually, conditions (i) and (ii) are referred to as the λ-LLC_1 and λ-LLC_2 conditions. Roughly speaking, the LLC condition rules out cusps and bubbles from the geometry of a metric space.

Remark 3.9. A space which satisfies the λ-LLC_1 condition is 4λ-bounded turning, and a space which is λ-bounded turning satisfies 2λ-LLC_1. The terminology "linearly locally connected" stems from the following fact. Let (X,d) satisfy the λ-LLC_1 condition, and let $x \in X$ and $r > 0$. Let $C(x)$ be the connected component of $B(x,r)$ containing x. Then $B(x,r/\lambda) \subseteq C(x) \subseteq B(x,r)$.

Definition 3.10. A metric space is Λ-linearly locally contractible, $\Lambda \geq 1$, if for all $a \in X$ and $r \leq \operatorname{diam}(X)/\Lambda$, the ball $B(a,r)$ is contractible inside the ball $B(a,\Lambda r)$.

Unfortunately, the term "linearly locally contractible" has not yet stabilized in the literature. Our definition is global in nature and agrees with the definitions given in [4] and [27]. The definition given in [14] is localized, and agrees with the following:

Definition 3.11. A metric space (X, d) is locally linearly locally contractible ($LLLC$) if for every compact subset $K \subseteq X$, there is a constant $\Lambda_K \geq 1$ and radius $R_K > 0$ such that for every point $x \in K$ and radius $0 < r \leq R_K$, the ball $B(x, r)$ is contractible inside the ball $B(x, \Lambda_K r)$.

As with Ahlfors regularity, it will be convenient to have a relative version as well.

Definition 3.12. A subset U of a metric space (X, d) is relatively Λ-locally linearly contractible if for all $x \in U$ and $0 < r \leq \operatorname{diam}(U)$, the ball $B_X(x, r)$ is contractible inside the ball $B_X(x, \Lambda r)$.

In certain situations, the LLC and linear local contractibility conditions are equivalent [4, Lemma 2.5]. We now localize this statement to show that the $LLLC$ condition implies a relative LLC condition for certain sets, quantitatively.

Definition 3.13. A subset U of a metric space (X, d) is relatively λ-LLC, $\lambda \geq 1$, if for all points $x \in U$ and $0 < r \leq \operatorname{diam}(U)$ the following conditions hold:

(i) for each pair of distinct points $y, z \in B_X(x, r)$, there is a continuum $\gamma \subseteq B_X(x, \lambda r)$ such that $y, z \in \gamma$,

(ii) if $B_X(x, r)$ is compactly contained in U, then for each pair of distinct points $y, z \in U \backslash B_X(x, r)$, there is a continuum $\gamma \subseteq U \backslash B_X(x, r/\lambda)$ such that $y, z \in \gamma$.

Proposition 3.14. *Suppose that (X, d) is a linearly locally contractible metric space homeomorphic to a connected topological n-manifold, $n \geq 2$, and let $K \subseteq X$ be compact. Let R_K and Λ_K be the constants associated with K by the linear local contractibility condition. If $U \subseteq K$ and $\operatorname{diam}(U) \leq R_K$, then U satisfies the first relative LLC condition with constant Λ_K. If in addition, U is connected and open in X, then U is relatively λ-LLC for any $\lambda > \Lambda_K$.*

As the proof of Proposition 3.14 is nearly identical to that of Lemma 2.4 in [4], we omit it.

Remark 3.15. We may consider alternate versions of bounded turning, LLC, and relative LLC where continua are replaced with arcs. These modified conditions are quantitatively equivalent to their original formulations in locally path connected spaces, as follows. It follows from [18, Theorems 3.15 and 3.30] that if $\gamma : [a, b] \to X$ is a path, then there is an arc α in X that connects $\gamma(a)$ and $\gamma(b)$ and is contained in $\operatorname{im} \gamma$. Thus, a simple covering argument shows that if (X, d) is locally path connected, and $E \subseteq X$ is a continuum that is contained in an open set $V \subseteq X$, then any pair of points $x, y \in E$ is contained in an arc in V.

3.4. **Quasisymmetric mappings.**

Quasisymmetric mappings first arose as the restrictions of quasiconformal mappings to the real line [2]. For the basic theory and applications of quasisymmetric mappings, see [30] and [13, Chapter 10].

Definition 3.16. A homeomorphism $f \colon X \to Y$ of metric spaces is called quasisymmetric if there exists a homeomorphism $\eta \colon [0, \infty) \to [0, \infty)$ such that for all triples $a, b, c \in X$ of distinct points,

$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} \leq \eta\left(\frac{d_X(a,b)}{d_X(a,c)}\right).$$

We will call the function η a distortion function of f; when η needs to be emphasized, we say that f is η-quasisymmetric. If f is η-quasisymmetric, then f^{-1} is also quasisymmetric with distortion function $(\eta^{-1}(t^{-1}))^{-1}$. Thus we say that metric spaces X and Y are quasisymmetric or quasisymmetrically equivalent if there is a quasisymmetric homeomorphism from X to Y.

The following result, due to Väisälä [31, Theorems 3.2, 3.10, 4.4, and 4.5], shows that the LLC condition is a quasisymmetric invariant.

Theorem 3.17 (Väisälä)**.** *If X is a λ-LLC metric space and $f\colon X \to Y$ is η-quasisymmetric, then Y is λ'-LLC for some λ' depending only on λ and η.*

The doubling condition is also quantitatively preserved by quasisymmetric mappings [30, Theorem 2.10]. A metric space that is quasisymmetrically equivalent to $\mathbb{S}^1$ is called a quasicircle. Tukia and Väisälä showed that the LLC and doubling conditions characterize quasicircles among all metric circles [30].

Theorem 3.18 (Tukia-Väisälä)**.** *A metric space (X, d) that is homeomorphic to $\mathbb{S}^1$ is a quasicircle if and only if (X, d) is doubling and LLC. The doubling and LLC constants can be chosen to depend only on the distortion function of the quasisymmetry, and vice-versa.*

Remark 3.19. It is an informative exercise to show that for any Jordan curve J in a metric space (X, d), the LLC condition may be restated in the following more intuitive fashion. Given any two distinct points $x, y \in J$, the set $J \backslash \{x, y\}$ consists of two disjoint arcs J_1 and J_2. The Jordan curve J is LLC if and only if there is some λ' such that for all pairs of distinct points $x, y \in J$,

$$\min\{\operatorname{diam} J_1, \operatorname{diam} J_2\} \leq \lambda' d(x, y). \tag{3.4}$$

The LLC constant of J and λ' depend only on each other. The condition (3.4) is often called the three-point condition. Theorem 3.18 implies that it may be used to characterize quasicircles as well.

A rectifiable path $\gamma\colon [a, b] \to X$ is said to be an l-chord-arc path, $l \geq 1$, if for every $s \leq t \in [a, b]$,

$$\operatorname{length}(\gamma|_{[s,t]}) \leq l d(\gamma(s), \gamma(t)).$$

Similarly, a continuous map $\gamma\colon \mathbb{S}^1 \to X$ is called an l-chord-arc loop if the following condition holds for all $\theta, \phi \in \mathbb{S}^1$. Let J_1 and J_2 be the unique subarcs of $\mathbb{S}^1$ such that $J_1 \cup J_2 = \mathbb{S}^1$ and $J_1 \cap J_2 = \{\theta, \phi\}$, then

$$\min\{\operatorname{length} \gamma|_{J_1}, \operatorname{length} \gamma|_{J_2}\} \leq l d(\gamma(\theta), \gamma(\phi)).$$

A chord-arc path or loop that is parameterized by arc length is an embedding, and the image is an arc or Jordan curve, respectively. Note that by Theorem 3.18 and the three-point characterization of the LLC condition given by (3.4), the image of a chord-arc loop is a quasicircle.

Ahlfors 2-regular and LLC metric spaces homeomorphic to a simply connected surface have been classified up to quasisymmetry [4], [32]. We will need the following statement, which is proven in manner similar to [32, Theorem 1.2 (iii)].

Theorem 3.20. *Let X be a metric space homeomorphic to the plane such that $\bar{X}$ is bounded, Ahlfors 2-regular, and LLC, and such that ∂X is a Jordan curve satisfying (3.4). Then X is quasisymmetrically equivalent to $\mathbb{D}^2$. The distortion function of the quasisymmetry can be chosen to depend only on the constants associated with the assumptions and the ratio* $\operatorname{diam} X/\operatorname{diam}\partial X$.

Proof. Throughout this proof, "the data" refers to the Ahlfors 2-regularity and LLC constants of $\bar{X}$, the constant associated to ∂X by (3.4), and the ratio $\operatorname{diam} X/\operatorname{diam}\partial X$. Let X' be the space obtained by gluing two copies of $\bar{X}$ together along ∂X. Then $\bar{X}$ embeds isometrically in to X', which is homeomorphic to $\mathbb{S}^2$. Furthermore, it is shown in [32, Section 5] that X' is Ahlfors 2-regular and LLC, with constants depending only on the data. Bonk and Kleiner's uniformization result for $\mathbb{S}^2$ [4, Theorem 1.1] implies that there is a quasisymmetric homeomorphism $f\colon X' \to \mathbb{S}^2$ with a distortion function that depends only on the data. By Theorem 3.17 and Remark 3.19, $f(\partial X)$ is an LLC Jordan curve in $\mathbb{S}^2$ with constants depending only on the data. The classical theory of conformal welding (cf. [21], [20]) now implies that there is a global quasisymmetric map $g\colon \mathbb{S}^2 \to \mathbb{S}^2$, with distortion depending only the data, such that $g \circ f(X) = \mathbb{D}^2$. □

4. Local uniformization

Theorem 1.1 states that if (X,d) is a locally Ahlfors 2-regular (Definition 3.2) and $LLLC$ (Definition 3.11) metric space homeomorphic to a surface, then each point $z \in X$ has a neighborhood which is quasisymmetrically equivalent to the disk. The following quantitative result immediately implies Theorem 1.1.

Theorem 4.1. *Let (X,d) be an LLLC and locally Ahlfors 2-regular metric space homeomorphic to a surface. Let K be a compact subset of X, and R_K, C_K, and Λ_K be the smallest radius and constants associated to K by the assumptions. Let z be an interior point of K and set*

$$R_0 = \min\{\max\{R \geq 0 : \bar{B}(z,R) \subseteq K\}, R_K\} > 0.$$

Then there exist constants $A_1, A_2 \geq 1$ depending only on C_K and Λ_K such that for all $0 < R \leq R_0/A_1$, there is a neighborhood Ω of z such that

(i) $B(z, R/A_2) \subseteq \Omega \subseteq B(z, A_2 R)$,
(ii) *there exists an η-quasisymmetric map $f\colon \Omega \to \mathbb{D}^2$, where η depends only on C_K and Λ_K.*

4.1. **Bounded turning and quasiarcs.** As discussed in the introduction, our proof of Theorem 4.1 requires that we first give a quasiconvexity result. To do so, we need a technical result similar to, but weaker than, the following theorem of Tukia [29]. Let $X \subseteq \mathbb{R}^n$ be endowed with the metric inherited from $\mathbb{R}^n$. If X is of bounded turning in itself, then any two points of X can be connected by a quasiarc, i.e., the quasisymmetric image of an interval. A short proof of this fact in a general setting has recently been published by Mackay [22].

Definition 4.2. Let $\epsilon > 0$ and $M \geq 1$. An arc α in a metric space (X,d) is an (ϵ, M)-quasiarc if for each pair of points $u, v \in \alpha$ with $d(u,v) \leq \epsilon$,

$$\operatorname{diam} \alpha[u,v] \leq M\epsilon.$$

Proposition 4.3. *Let (X,d) be a locally path connected metric space, and $z \in X$. Suppose that there is a radius $R > 0$ and constants Q, D, and λ such that $B(z,R)$ is of λ-bounded turning in X, and has relative Assouad dimension at most Q with constant D. Then there are constants $M, N, c \geq 1$, all depending only on Q, D, and λ, with the following property. For all pairs of points $x, y \subseteq B(z, R/c)$ and all $0 < \epsilon < d(x,y)$, there is an (ϵ, M)-quasiarc connecting x to y that is contained in $B(x, Nd(x,y))$.*

For the proof, we need a lemma regarding the extraction of an arc from the image of a path (see Remark 3.15). In general, there is not a unique way to do so. However, in the case that the path is a concatenation of embeddings, there is a canonical choice.

Proposition 4.4. *Let (X,d) be a metric space and $n \in \mathbb{N}$. For $i = 0, \ldots, n$, let $\gamma_i \colon [a_i, b_i] \to X$ be an embedding. For $i = 0, \ldots, n-1$, assume that $\gamma_i(b_i) = \gamma_{i+1}(a_{i+1})$. Then there is an arc α connecting $\gamma_0(a_0)$ to $\gamma_n(b_n)$ with the following property: if $u, v \in \alpha$, then there are indices $i, j \in \{0, \ldots, n\}$ such that $u \in \operatorname{im}(\gamma_i)$, $v \in \operatorname{im}(\gamma_j)$, and $\alpha[u,v]$ is contained in $\bigcup \operatorname{im}(\gamma_l)$, where l ranges over indices between i and j, inclusively.*

Proof. We construct the arc via an inductive process. Let $i_0 = 0$ and $a'_0 = a_0$. Let i_1 be the largest index $i \in \{1, \ldots, n\}$ such that $\operatorname{im}(\gamma_0) \cap \operatorname{im}(\gamma_i) \neq \emptyset$; this is well defined since $\gamma_0(b_0) = \gamma_1(a_1)$. Set

$$b'_0 = \gamma_0^{-1} \circ \gamma_{i_1} \left(\max\{t \in [a_{i_1}, b_{i_1}] : \gamma_{i_1}(t) \in \operatorname{im} \gamma_0\} \right).$$

Now assume that $k \geq 1$, and that the indices $i_{k-1} < i_k \in \{1, \ldots, n\}$ and the interval $[a'_{i_{k-1}}, b'_{i_{k-1}}] \subseteq [a_{i_{k-1}}, b_{i_{k-1}}]$ are defined and satisfy

$$\gamma_{i_{k-1}}(b'_{i_{k-1}}) \in \gamma_{i_k}.$$

Set

$$a'_{i_k} = \gamma_{i_k}^{-1}(\gamma_{i_{k-1}}(b'_{k-1})) \in [a_{i_k}, b_{i_k}].$$

If $i_k = n$, set $b'_{i_k} = b_n$ and stop. If not, let i_{k+1} be the largest index $i \in \{i_k + 1, \ldots, n\}$ such that

$$\gamma_{i_k}([a'_{i_k}, b_{i_k}]) \cap \operatorname{im}(\gamma_i) \neq \emptyset,$$

and define

$$b'_{i_k} = \gamma_{i_k}^{-1} \circ \gamma_{i_{k+1}} \left(\max\{t \in [a_{i_{k+1}}, b_{i_{k+1}}] : \gamma_{i_{k+1}}(t) \in \operatorname{im} \gamma_{i_k}\} \right).$$

Since $i_k < i_{k+1}$, this process stops after finitely many steps. Let $i_m = n$ be the final index. Define α to be the image of the concatenation of $\gamma_{i_k}|_{[a'_{i_k}, b'_{i_k}]}$ for $k = 0, \ldots, m$. Then α is an arc with the desired properties. □

We will also need a notion of a discrete path. For $\epsilon > 0$, an ϵ-chain connecting points x and y of X is defined to be a sequence of points $x = x_0, x_1, \ldots, x_n = y$ in X such that $d(x_i, x_{i+1}) \leq \epsilon$ for each $i = 0, \ldots, n-1$. We will often denote ϵ-chains using bold face, e.g., $\mathbf{x} = x_0, \ldots, x_n$. In a connected space, any two points may be connected by an efficient chain.

Lemma 4.5. *Let (X,d) be a connected metric space and $\epsilon > 0$. For any pair of points $x, y \in X$, there is an ϵ-chain $x_0, \ldots, x_n$ connecting x to y that contains an ϵ-separated set of cardinality at least $n/2$.*

Proof. For any $z \in X$, let

$$S(z) := \bigcup \{w \in X : \text{ there exists an } \epsilon\text{-chain from } z \text{ to } w\}.$$

Then $S(z)$ is an open set, and if $S(z) \cap S(w) \neq \emptyset$, then $S(z) = S(w)$. By connectedness, we see that $S(x) = X$. If $x_0, \ldots, x_n$ is the ϵ-chain from x to y of minimal cardinality, then $d(x_i, x_j) \geq \epsilon$ for all $i = 0, \ldots, n-2$ and $j \geq i+2$. This implies that the set of even-indexed points in the chain is ϵ-separated. □

Proof of Proposition 4.3. The basic idea is the following. Take an ϵ-chain of minimal cardinality connecting x to y. We may use the bounded turning condition to connect consecutive points of the chain. The resulting concatenation contains an (ϵ, M)-quasiarc connecting x to y, for otherwise we may find a shorter ϵ-chain. Unfortunately, it is not true in general that this arc will be contained in a ball around x with controlled radius. To overcome this, we introduce a "score" function on ϵ-chains that balances distance from x with cardinality.

We now begin the formal proof. As per Remark 3.15, we may assume with out loss of generality that the bounded turning condition provides arcs rather than arbitrary continua.

Set

$$c = 1 + 8\lambda + 2(4D)^{\frac{1}{2Q}}(4\lambda)^{\frac{1}{2}}.$$

Let $x, y \in B(z, R/c)$, and set $d(x,y) = r$. The bounded turning condition provides an arc γ connecting x to y with $\operatorname{diam} \gamma \leq \lambda r$. Let $\epsilon < r$. For any ϵ-chain $\mathbf{w}$ in X, define the ϵ-score function

$$\sigma_\epsilon(\mathbf{w}) = \sum_{w \in \mathbf{w}} 1 + \left(\frac{\operatorname{dist}(w, \gamma)}{\epsilon} \right)^{2Q}.$$

As $\gamma \subseteq B(x, 2\lambda r)$ and $2\lambda r < 4\lambda R/c < R$, the arc γ may be covered by at most $D(4\lambda r/\epsilon)^Q$ balls of radius $\epsilon/2$. By Lemma 4.5, there is an ϵ-chain $\mathbf{w} \subseteq \gamma$ connecting x to y containing an ϵ-separated set of cardinality at least $\operatorname{card}(\mathbf{w})/2$. Since ϵ-separated points cannot be contained in a single $(\epsilon/2)$-ball, we see that

$$\operatorname{card} \mathbf{w} \leq 2D(4\lambda r/\epsilon)^Q.$$

Let $\mathcal{A}_\epsilon(x,y)$ be the set of all ϵ-chains in X connecting points $x, y \in X$. Let $\{z_0, \ldots, z_n\} = \mathbf{z} \in \mathcal{A}_\epsilon(x,y)$ be such that

$$\sigma_\epsilon(\mathbf{z}) \leq \inf_{\mathbf{x} \in \mathcal{A}_\epsilon(x,y)} \sigma_\epsilon(\mathbf{x}) + 1.$$

Note that

$$\inf_{\mathbf{x} \in \mathcal{A}_\epsilon(x,y)} \sigma_\epsilon(\mathbf{x}) \geq 2,$$

and so we also have

$$\sigma_\epsilon(\mathbf{z}) \leq 2 \inf_{\mathbf{x} \in \mathcal{A}_\epsilon(x,y)} \sigma_\epsilon(\mathbf{x}).$$

Set

$$H := \max_{i=0,\ldots,n} \operatorname{dist}(z_i, \gamma).$$

Then

$$\left(\frac{H}{\epsilon}\right)^{2Q} \leq \sigma_\epsilon(\mathbf{z}) \leq 2\sigma_\epsilon(\mathbf{w}) \leq 4D\left(\frac{4\lambda r}{\epsilon}\right)^Q.$$

As a result, we have

$$H \leq (4D)^{1/(2Q)}(4\lambda r\epsilon)^{1/2}.$$

Since $r\epsilon < (2R/c)^2$, we have for $i = 0, \dots, n$,

$$d(z, z_i) \leq d(z, x) + \operatorname{diam}\gamma + H < R/c + 2\lambda R/c + (4D)^{1/(2Q)}(4\lambda)^{1/2} 2R/c < R.$$

We may now apply the bounded turning condition, providing for each $i = 0, \dots, n-1$ an embedding $\gamma_i\colon [0,1] \to X$ that connects z_i to z_{i+1} with $\operatorname{diam}(\operatorname{im}\gamma_i) \leq \lambda\epsilon$.

Note that if $p \in \operatorname{im}\gamma_i$ for some $i = 0, \dots, n-1$, then

$$d(x, p) \leq \operatorname{dist}(p, \mathbf{z}) + H + \operatorname{diam}(\gamma) \leq \lambda\epsilon + (4D)^{\frac{1}{2Q}}(4\lambda r\epsilon)^{\frac{1}{2}} + \lambda r.$$

Since $\epsilon < r$, this implies that $\operatorname{im}\gamma_i \subseteq B(x, Nr)$ for $N = 2\lambda + (4D)^{\frac{1}{2Q}}(4\lambda)^{\frac{1}{2}}$.

We now make a claim which will quickly imply the desired result. Let $i, j \in \{0, \dots, n-1\}$, and suppose that $u \in \operatorname{im}\gamma_i$ and $v \in \operatorname{im}\gamma_j$ are points such that $d(u, v) \leq \epsilon$. Then there is an integer M_0, depending only on Q, D, and λ, such that $|j - i| \leq M_0$.

Suppose that the claim is true. We may extract an arc α connecting x to y from the image of the concatenation $\gamma_0 \cdot \dots \cdot \gamma_{n-1}$, as in Proposition 4.4. As $\operatorname{im}\gamma_i \subseteq B(x, Nr)$ for $i = 0, \dots, n-1$, we have that $\alpha \subseteq B(x, Nr)$. We now show that α is an $(\epsilon, M_0\lambda)$-quasiarc. Let $u, v \in \alpha$ be such that $d(u, v) \leq \epsilon$. We may find indices $i, j \in \{0, \dots, n-1\}$ such that $u \in \operatorname{im}\gamma_i$, $v \in \operatorname{im}\gamma_j$, and $\alpha[u, v]$ is contained in $\bigcup \operatorname{im}(\gamma_l)$, where l ranges over the indices between i and j, inclusively. Without loss of generality, assume $i \leq j$. The claim yields

$$\operatorname{diam}(\alpha[u, v]) \leq \operatorname{diam}\left(\bigcup_{l=i}^{j} \operatorname{im}\gamma_l\right) \leq \sum_{l=i}^{j} \operatorname{diam}(\operatorname{im}(\gamma_l)) \leq M_0\lambda\epsilon,$$

as desired.

We now prove the claim. Let $i, j \in \{0, \dots, n-1\}$, and suppose that $u \in \operatorname{im}\gamma_i$ and $v \in \operatorname{im}\gamma_j$ are points such that $d(u, v) \leq \epsilon$. Without loss of generality, we assume that $i \leq j$. Note that as $x \in B(z, R/c)$,

$$\operatorname{im}(\gamma_i) \subseteq B(x, Nr) \subseteq B(z, (2N+1)R/c) \subseteq B(z, R),$$

and so there is a cover of $\operatorname{im}\gamma_i$ by no more than $D(2\lambda)^Q$ balls of radius $\epsilon/2$. The same holds for $\operatorname{im}\gamma_j$. Applying Lemma 4.5 to $\operatorname{im}\gamma_i$ and $\operatorname{im}\gamma_j$ provides ϵ-chains $\mathbf{w}_u$ and $\mathbf{w}_v$ connecting z_i to u and v to z_{j+1}, each of cardinality no greater than $2D(2\lambda)^Q$. It follows that

$$\mathbf{w} := \{z_0, \dots, z_{i-1}\} \cup \mathbf{w}_u \cup \mathbf{w}_v \cup \{z_{j+2}, \dots, z_n\}$$

is an ϵ-chain. See Figure 1.

We now use the inequality

$$\sigma_\epsilon(\mathbf{z}) \leq \sigma_\epsilon(\mathbf{w}) + 1.$$

Canceling at the points where $\mathbf{z}$ and $\mathbf{w}$ agree, this inequality simplifies to

$$\sum_{l=i}^{j+1} 1 + \left(\frac{\operatorname{dist}(z_l, \gamma)}{\epsilon}\right)^{2Q} \leq \sigma_\epsilon(\mathbf{w}_u \cup \mathbf{w}_v) + 1. \tag{4.1}$$

Let $\Theta = \operatorname{dist}(z_i, \gamma)/\epsilon$, $m = j - i + 1$, and $A = \operatorname{card}(\mathbf{w}_u \cup \mathbf{w}_v) - 1$. Then A depends only on Q, D, and λ. Note that if a and b are consecutive points of an ϵ-chain, then

$$\left|\frac{d(a, \gamma)}{\epsilon} - \frac{d(b, \gamma)}{\epsilon}\right| \leq 1.$$

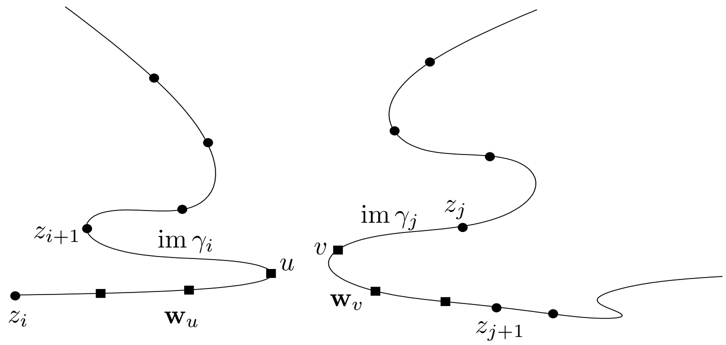

FIGURE 1. The shortcut chains $\mathbf{w}_u$ and $\mathbf{w}_v$

Now (4.1) implies

$$\sum_{l=0}^{m} 1 + (\max\{\Theta - l, 0\})^{2Q} \leq 1 + \sum_{l=0}^{A} 1 + (\Theta + l)^{2Q}. \tag{4.2}$$

In order to show that m, and hence $j - i$, is bounded above by a constant depending only on Q, λ, and D, we must analyze a few cases. Let

$$\Theta_0 := \sup\left\{\Theta : 1 + (A+1)(1 + (\Theta + A)^{2Q}) - \frac{\Theta^{2Q+1} - (\Theta/2)^{2Q+1}}{2Q+1} \geq 0\right\}.$$

Then Θ_0 is finite and depends only on Q, D, and λ.

Case 1: $\Theta \leq \Theta_0$. By (4.2), we have

$$m \leq \sum_{l=0}^{m} 1 + (\max\{\Theta - l, 0\})^{2Q} \leq 1 + \sum_{l=0}^{A} 1 + (\Theta + l)^{2Q} \leq 1 + (A+1)(1 + (\Theta_0 + A)^{2Q}). \tag{4.3}$$

This provides the desired bound.

Case 2: $\Theta > \Theta_0$ and $m \geq \Theta/2$. We have

$$\sum_{l=0}^{m} 1 + (\max\{\Theta - l, 0\})^{2Q} \geq m + \sum_{l=0}^{\lfloor \Theta/2 \rfloor} (\Theta - l)^{2Q} \geq m + \int_0^{\lfloor \Theta/2 \rfloor + 1} (\Theta - l)^{2Q} dl.$$

This, combined with (4.2), shows that

$$m + \frac{\Theta^{2Q+1} - (\Theta/2)^{2Q+1}}{2Q+1} \leq 1 + (A+1)(1 + (\Theta + A)^{2Q}).$$

By the definition of Θ_0, this yields that $m < 0$, a contradiction.

Case 3: $\Theta > \Theta_0$ and $m \leq A$. As A depends only on Q, D, and λ, we already have the desired bound.

Case 4: $\Theta > \Theta_0$ and $A < m \leq \Theta/2$. In this case,

$$m\left(\frac{\Theta}{2}\right)^{2Q} \leq \sum_{l=0}^{m} 1 + \max\left\{(\Theta - l)^{2Q}, 0\right\}.$$

Noting that $A > 1$ by definition, we see that

$$1 + \sum_{l=0}^{A} 1 + (\Theta + l)^{2Q} \leq 1 + (A+1)(1 + (\Theta + A)^{2Q}) \leq 8A(2\Theta)^{2Q}.$$

From these inequalities and (4.2), we see that $m \leq 8A(4)^{2Q}$, as desired.

Combining these cases shows that m is bounded above by a constant depending only on Q, λ, and D, which completes the proof of the claim. □

4.2. **Quasiconvexity.**

Theorem 4.6. *Let (X, d) be a metric space, and $z \in X$. Suppose that $U \subseteq X$ is a neighborhood of z homeomorphic to the plane $\mathbb{R}^2$, and that there are constants $C, \Lambda, M, N \geq 1$ and a radius $R > 0$ such that*

(i) *$\bar{B}(z, 4NR)$ is a compact subset of U,*
(ii) *U is relatively Ahlfors 2-regular with constant C (see Definition 3.3),*
(iii) *U is relatively Λ-linearly locally contractible (see Definition 3.12),*
(iv) *for all $x, y \in U$ and $0 < \epsilon < d(x, y)$, there is an (ϵ, M)-quasiarc connecting x to y inside of $B(x, Nd(x, y))$ (see Definition 4.2).*

Then there exists a constant $L \geq 1$ depending only on C, Λ, M, and N such that each pair of points $x, y \in B(z, R)$ may be joined by an arc γ in X such that $\operatorname{length}(\gamma) \leq Ld(x, y)$.

Remark 4.7. For any pair of points $x, y \in U$, setting $\epsilon = d(x, y)/2$ in assumption (iv) above provides *some* arc connecting x to y inside the ball $B(x, Nd(x, y))$. Of course, a similar connection (with a different constant) is provided by assumption (iii), as in Proposition 3.14.

Propositions 3.6,3.14, and 4.3 now yield the following corollary.

Corollary 4.8. *A locally Ahlfors 2-regular, and LLLC metric space homeomorphic to a surface is locally quasiconvex.*

We recall that for any continuous map $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ and point $z \in \mathbb{R}^2 \setminus \operatorname{im}\gamma$, the index $\operatorname{ind}(\gamma, z)$ of γ with respect to z is an integer which indicates the number of times γ "wraps around z", taking orientation into account. For a full definition and description, see for example [10, Chapter 4.2]. The fundamental property of the index is that it behaves well under homotopies. If $H\colon \mathbb{S}^1 \times [0,1] \to \mathbb{R}^2$ is continuous, and $z \notin \operatorname{im} H$, then $\operatorname{ind}(H(\cdot, 0), z) = \operatorname{ind}(H(\cdot, 1), z)$. Moreover, the index is additive under concatenation: if $\gamma_1, \gamma_2\colon \mathbb{S}^1 \to \mathbb{R}^2$ are loops with a common basepoint and $z \in \mathbb{R}^2$ is a point not in the image of either loop, then $\operatorname{ind}(\gamma_1 \cdot \gamma_2, z) = \operatorname{ind}(\gamma_1, z) + \operatorname{ind}(\gamma_2, z)$. Finally, the function $\operatorname{ind}(\gamma, \cdot)$ is constant on connected components of $\mathbb{R}^2 \setminus \operatorname{im}\gamma$.

Given a Jordan curve J in the plane, we may find a parameterization γ of J such that $\operatorname{ind}(\gamma, z) = 1$ for all $z \in \operatorname{inside}(J)$ and $\operatorname{ind}(\gamma, z) = 0$ for all $z \in \mathbb{R}^2 \setminus (J \cup \operatorname{inside}(J))$. If the image of a continuous map $\gamma\colon \mathbb{S}^1 \to \mathbb{R}^2$ is an arc, then $\operatorname{ind}(\gamma, z) = 0$ for all $z \notin \operatorname{im}(\gamma)$.

Assumption (iii) of Theorem 4.6 provides convenient criteria for showing that there is a controlled homotopy between certain loops. This trick will play a role in the proof of Theorem 4.6 as well as the rest of the proof of Theorem 4.1.

Lemma 4.9. *Let (X, d) be a metric space, and let $\Lambda \geq 1$ and $\delta > 0$. Suppose that $U \subseteq X$ is a subset such that for all $x \in U$ and $0 < r \leq 2\delta(\Lambda + 1)$ the ball $B(x, r)$ is* *contractible inside the ball $B(x, \Lambda r)$. Let α and β be continuous maps of $\mathbb{S}^1$ into U. If there exists a cyclically ordered set $\{\theta_1, \dots, \theta_n\} \subseteq \mathbb{S}^1$ such that $d(\alpha(\theta_i), \beta(\theta_i)) \leq \delta$ for $i = 1, \dots, n$, and*

$$\max\{\operatorname{diam}(\alpha([\theta_i, \theta_{i+1}])), \operatorname{diam}(\beta([\theta_i, \theta_{i+1}]))\} \leq \delta$$

for $i = 1, \dots, n$, $\bmod n$, then there is a homotopy $H\colon \mathbb{S}^1 \times [0,1] \to \mathcal{N}_{2\Lambda\delta(\Lambda+1)}(\operatorname{im}\beta)$ such that $H(\cdot, 0) = \alpha$ and $H(\cdot, 1) = \beta$.

Proof. The contractibility assumption implies that for every pair of points $x, y \in U$ with $d(x,y) < \delta(\Lambda+1)$, there is a path connecting x to y inside $B(x, 2\Lambda d(x,y))$.

Throughout this proof, consider all indices modulo n. Fix $i \in \{1, \dots, n\}$, and define $a_i = \alpha(\theta_i)$ and $b_i = \beta(\theta_i)$. We may find a path $\gamma_i\colon [0,1] \to B(b_i, 2\Lambda\delta)$ such that $\gamma_i(0) = a_i$ and $\gamma_i(1) = b_i$.

Define a subset L_i of $\mathbb{S}^1 \times [0,1]$ by $L_i = [\theta_i, \theta_{i+1}] \times [0,1]$. The strip L_i is bounded by the curve

$$l_i := ([\theta_i, \theta_{i+1}] \times \{0\}) \cup (\{\theta_{i+1}\} \times [0,1]) \cup ([\theta_i, \theta_{i+1}] \times \{1\}) \cup (\{\theta_i\} \times [0,1]).$$

Define a continuous map $g_i : l_i \to X$ by

$$g_i(\theta, s) = \begin{cases} \alpha(\theta) & (\theta, s) \in [\theta_i, \theta_{i+1}] \times \{0\}, \\ \gamma_{i+1}(s) & (\theta, s) \in \{\theta_{i+1}\} \times [0,1], \\ \beta(\theta) & (\theta, s) \in [\theta_i, \theta_{i+1}] \times \{1\}, \\ \gamma_i(s) & (\theta, s) \in \{\theta_i\} \times [0,1]. \end{cases}$$

As

$$\gamma_i([0,1]) \subseteq B(b_i, 2\Lambda\delta), \quad \text{and} \quad \gamma_{i+1}([0,1]) \subseteq B(b_{i+1}, 2\Lambda\delta),$$

and

$$\alpha([\theta_i, \theta_{i+1}]) \subseteq B(a_i, 2\delta), \quad \text{and} \quad \beta([\theta_i, \theta_{i+1}]) \subseteq B(b_i, 2\delta),$$

we see that $\operatorname{im}(g_i) \subseteq B(b_i, 2\delta(\Lambda+1))$. The contractibility assumption now provides a homotopy

$$H_i\colon B(b_i, 2\delta(\Lambda+1)) \times [0,1] \to B(b_i, 2\Lambda\delta(\Lambda+1))$$

such that $H_i(\cdot, 0)$ is the identity map and $H_i(\cdot, 1)$ is a constant map.

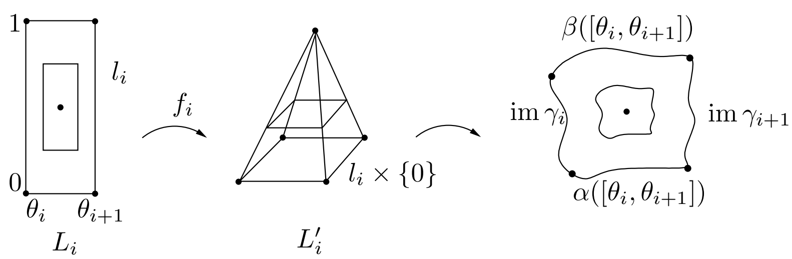

FIGURE 2. The homotopy trick

Let L_i' be the quotient of $l_i \times [0,1]$ obtained by identifying all points in $l_i \times \{1\}$. Then there is a homeomorphism $f_i\colon L_i \to L_i'$ which maps l_i to the copy of $l_i \times \{0\}$ in L_i'. Denote

$$f_i(\theta, s) = [l(\theta, s), t(\theta, s)] \in L_i'.$$

Define $G_i\colon L_i \to X$ by

$$G_i(\theta, s) = H_i(g_i(l(\theta, s)), t(\theta, s)) \subseteq B(b_i, 2\Lambda\delta(\Lambda + 1)).$$

See Figure 2. This is well-defined and continuous because $H_i(\cdot, 1)$ is a constant map. Furthermore, $G_i|_{l_i} = g_i$, since $H(\cdot, 0)$ is the identity mapping and f_i maps l_i to $l_i \times \{0\}$. As a result, G_i agrees with G_{i+1} on $L_i \cap L_{i+1} = \{\theta_{i+1}\} \times [0, 1]$. Thus we may form a continuous map $G\colon \mathbb{S}^1 \times [0, 1] \to X$ by setting $G = G_i$ on L_i. From the definitions of G_i and g_i, we see that $G(\cdot, 0) = \alpha$ and $G(\cdot, 1) = \beta$, as desired. □

Our main tool in the proof of Theorem 4.6 is the following consequence of a co-area formula for Lipschitz maps of metric spaces. See [11, Theorem 2.10.25] for a proof and discussion.

Theorem 4.10. *Let (X, d) be a metric space, and let $E \subseteq X$ be a continuum. For $t > 0$, set*

$$E_t := \{x \in X : \operatorname{dist}(x, E) < t\} \quad \text{and} \quad L_t := \{x \in X : \operatorname{dist}(x, E) = t\}.$$

There exists a universal constant ω such that if $T \geq 0$ and

$$\int_0^T \mathcal{H}^1(L_t)\ dt \leq \omega \mathcal{H}^2(E_T \cup L_T).$$

A continuum E in a metric space need not be locally connected, and hence is not necessarily arc-connected. The additional condition that $\mathcal{H}^1(E) < \infty$ provides this property. Moreover, the arcs can be chosen so that their length is majorized by the $\mathcal{H}^1(E)$. See [27, Section 15] for a proof of the following well-known result.

Proposition 4.11. *Let E be a continuum in a metric space (X, d) such that $\mathcal{H}^1(E) < \infty$, and let $x, y \in E$. Then there exists an embedding $\gamma\colon [0, 1] \to E$ connecting x to y such that* $\operatorname{length} \gamma \leq \mathcal{H}^1(E)$.

We will also need some facts from elementary topology. The following statement follows from the compactness of $\mathbb{S}^1$ and the fact that a continuous map on a compact set is uniformly continuous.

Proposition 4.12. *Let (X, d) be a metric space, and $\alpha : \mathbb{S}^1 \to X$ a continuous map. For each $t > 0$, there exists a finite subset $\{\theta_1, \ldots, \theta_n\} \subseteq \mathbb{S}^1$ in cyclic order such that* $\operatorname{diam}(\alpha([\theta_i, \theta_{i+1}])) \leq t$ *for* $i = 1, \ldots, n \bmod n$.

By Schoenflies' Theorem, a Jordan curve in the plane is the boundary of its inside. The inside of a Jordan curve can be used to approximate a compact, connected subset of the plane [10, Page 347].

Proposition 4.13. *Let K be a compact and connected subset of $\mathbb{R}^2$. For any open set $V \supseteq K$, there is a Jordan curve α such that $K \subseteq \operatorname{inside} \alpha$ and $\alpha \subseteq V$.*

The final topological fact we need is also well-known. Here we consider the sphere $\mathbb{S}^2$ as the one point compactification of $\mathbb{R}^2$, with the added point labeled ∞.

Proposition 4.14. *Let V be an open and connected subset of $\mathbb{R}^2$ such that $\mathbb{R}^2 \backslash V$ is a continuum. Then $V \cup \{\infty\} \subseteq \mathbb{S}^2$ is homeomorphic to the plane. Moreover, the topological boundary of V in $\mathbb{R}^2$ is connected.*

Proof. The first statement is an immediate corollary of Proposition 4.27, which is proven independently. The second statement is elementary; it follows, for example, from [10, Exercise 1.5] and the Riemann mapping theorem. □

Proof of Theorem 4.6. Let $x, y \in B(z, R)$, and set $r = d(x, y)$, and

$$\epsilon = \frac{r}{32MN\Lambda(\Lambda+1)}. \tag{4.4}$$

Since $\epsilon < r$, there is an (ϵ, M)-quasiarc γ connecting x to y inside of $B(x, Nr) \subseteq U$.

Let $\underline{\gamma}\colon [0,1] \to X$ be an embedding parameterizing γ with $\underline{\gamma}(0) = x$ and $\underline{\gamma}(1) = y$. Define

$$a = \max\{s \in [0,1] : \underline{\gamma}(s) \in \bar{B}(x, r/8N)\} \quad \text{and} \quad b = \min\{s \in [a,1] : \underline{\gamma}(s) \in \bar{B}(y, r/8N)\}.$$

Define $E = \underline{\gamma}([a,b])$. Since $d(x,y) > r/8N$, we see that $a < 1$. As γ is connected, this implies that $d(\gamma(a), x) = r/8N$, and so $\operatorname{dist}(x, E) = r/8N$. Similarly $\operatorname{dist}(y, E) = r/8N$. For $t > 0$, set

$$E_t := \{w \in U : \operatorname{dist}(w, E) \le t\} \quad \text{and} \quad L_t := \{w \in U : \operatorname{dist}(w, E) = t\}.$$

Then

$$E_{r/8N} \cup L_{r/8N} \subseteq B(x, 2Nr).$$

See Figure 3.

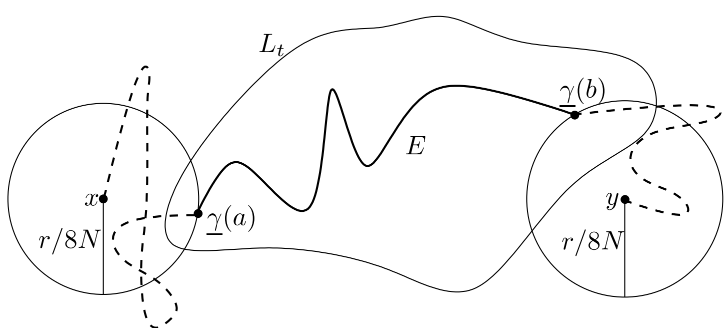

FIGURE 3. Connecting $B(x, r/8N)$ and $B(y, r/8N)$

Since $x \in U$, and $2Nr \le 4NR \le \operatorname{diam}(U)$, assumptions (i) and (ii) along with Theorem 4.10 yield

$$\int_0^{r/8N} \mathcal{H}^1(L_t)\, dt \le \omega \mathcal{H}^2(B(x, 2Nr)) \le 4C\omega N^2 r^2.$$

From this we may estimate that for any $s > 8N$,

$$|\{t \in [0, r/s] : \mathcal{H}^1(L_t) < 8C\omega N^2 sr\}| \ge \frac{r}{2s}. \tag{4.5}$$

We claim that there is a constant $s_0 > 8N$, depending only on Λ, M, and N, such that for all $t \in [0, r/s_0]$, there is a closed and connected subset of L_t that intersects both $B(x, r/4)$ and $B(y, r/4)$.

Suppose that the claim is true. The measure estimate (4.5) ensures that there is some $t_0 \in [0, r/s_0]$ such that the level set L_{t_0} satisfies $\mathcal{H}^1(L_{t_0}) \le 8C\omega N^2 s_0 r$. By assumption (i), the set L_{t_0} is compact, and so the closed and connected subset of L_{t_0} intersecting both $B(x, r/4)$ and $B(y, r/4)$ is a continuum of controlled $\mathcal{H}^1$-measure. Proposition 4.11 now provides a path of length no greater than Lr connecting the balls, where L depends only on C, Λ, M, and N. The desired arc connecting x to

y is now easily constructed using an inductive process and arc-extraction; see [5, Lemma 3.4] for more details.

We proceed with the proof of the claim. Let

$$s_0 = 5r/\epsilon = 160\Lambda MN(\Lambda + 1),$$

and let $t \in [0, r/s_0]$. Consider the set

$$F_t := \{w \in U : d(w, E) > t\}.$$

As $d(x, E) = r/8N > t$, the point x must belong to some component A of F_t.

By assumption (i), the set $U \backslash F_t = E_t \cup L_t$ is compact. As U is homeomorphic to the plane, this implies that there is a unique component of F_t whose closure is not a compact subset of U. We show that A must be this component. Towards a contradiction, suppose that $\mathrm{cl}_U\, A$ is a compact subset of U. Then by Proposition 4.13, there exists a embedding $\alpha : \mathbb{S}^1 \to U$ such that $\mathrm{im}(\alpha) \subseteq \mathcal{N}_{t/N}(A)$ and A is contained in the inside of the Jordan curve $\mathrm{im}(\alpha)$.

By Proposition 4.12, there is a cyclically ordered set $\{\theta_1, \ldots, \theta_n\} \subseteq \mathbb{S}^1$ such that

$$\mathrm{diam}(\alpha([\theta_i, \theta_{i+1}])) \leq t \tag{4.6}$$

for $i = 1, \ldots, n$, mod n. Without loss of generality, we may assume that

$$0 = \theta_1 < \ldots < \theta_n < 2\pi.$$

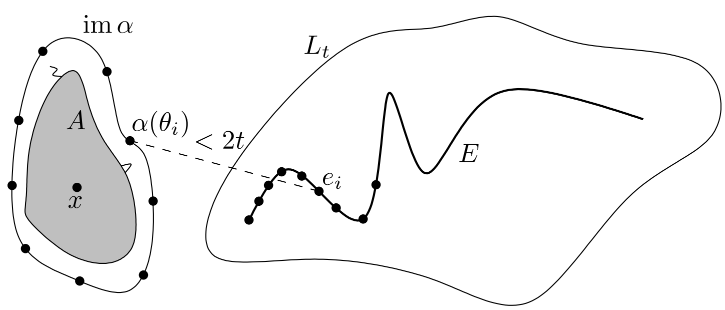

FIGURE 4. The points $\alpha(\theta_i)$ and their partners e_i

If $w \in \mathrm{im}(\alpha)$, then there is a point $a \in A$ such that $d(w, a) < t/N$. By Remark 4.7 there is an arc connecting w to a inside $B(w, t)$. As $a \in A$, $w \notin A$, and A is a component of F_t, this path must intersect L_t. This implies that $\mathrm{dist}(w, E) < 2t$. Thus for each parameter θ_i, we may find a point $e_i \in E$ such that $d(\alpha(\theta_i), e_i) < 2t$. Note that the set $\{e_1, \ldots, e_n\}$ need not be linearly ordered with respect to the given parameterization $\underline{\gamma}$. See Figure 4. For $i = 1, \ldots, n$, set $t_i = \underline{\gamma}^{-1}(e_i) \in [a, b]$. For $i = 1, \ldots, n-1$, define $\rho_i : [\theta_i, \theta_{i+1}] \to \mathbb{R}$ to be the unique linear map satisfying $\rho_i(\theta_i) = t_i$ and $\rho_i(\theta_{i+1}) = t_{i+1}$, and set $\rho_n : [0, 2\pi - \theta_n] \to \mathbb{R}$ to be the unique linear map satisfying $\rho_n(0) = t_n$ and $\rho_n(2\pi - \theta_n) = t_1$. Define $f \colon \mathbb{S}^1 \to E$ by

$$f(\theta) = \begin{cases} \underline{\gamma} \circ \rho_i(\theta) & \theta \in [\theta_i, \theta_{i+1}], i = 1, \ldots, n-1, \\ \underline{\gamma} \circ \rho_n(\theta - \theta_n) & \theta \in [\theta_n, 2\pi). \end{cases}$$

Then for $i = 1, \ldots, n$ $f|_{[\theta_i, \theta_{i+1}]}$ is an embedding parameterizing $E[e_i, e_{i+1}]$. Thus $\operatorname{im} f \subseteq E$. Since E is an arc and $x \notin \operatorname{im} f$, this implies that $\operatorname{ind}(f, x) = 0$. On the other hand, $\operatorname{ind}(\alpha, x) \neq 0$, since $x \in A \subseteq \operatorname{inside}(\operatorname{im} \alpha)$.

To reach a contradiction, we will apply Lemma 4.9 to α and f, using $\delta = M\epsilon$. By the definition of ϵ and assumption (i), we have

$$2M\epsilon(\Lambda + 1) = \frac{r}{16\Lambda N} \leq \frac{R}{8\Lambda N} \leq \operatorname{diam}(U).$$

Thus by assumption (iii), each ball $B(x, \rho)$ with $x \in U$ and $\rho \leq 2M\epsilon(\Lambda + 1)$ is contractible inside $B(x, \Lambda\rho)$. This verifies the first hypothesis of Lemma 4.9. Consider the points $\{\alpha(\theta_1), \ldots, \alpha(\theta_n)\} \subseteq \operatorname{im} \alpha$ and $\{e_1, \ldots, e_n\} \subseteq \operatorname{im} f$. We have already seen that $d(\alpha(\theta_i), e_i) < 2t < \epsilon$ for $i = 1, \ldots, n$, and that

$$\operatorname{diam}(\alpha([\theta_i, \theta_{i+1}])) \leq t \leq \epsilon$$

for $i = 1, \ldots, n$, $\bmod n$. This implies that $d(e_i, e_{i+1}) < 5t \leq \epsilon$ for $i = 1, \ldots, n$, $\bmod n$. Since γ is an (ϵ, M)-quasiarc,

$$\operatorname{diam}(f([\theta_i, \theta_{i+1}])) \leq M\epsilon.$$

These statements verify the remaining hypotheses of Lemma 4.9, and we conclude that f is homotopic to α inside $\mathcal{N}_{2M\Lambda\epsilon(\Lambda+1)}(\operatorname{im} f)$. Since $\operatorname{im} f \subseteq E$, our choice of ϵ and the fact that $\operatorname{dist}(x, E) = r/8N$ show that the tracks of the homotopy do not hit x. This contradicts that $\operatorname{ind}(f, x) \neq \operatorname{ind}(\alpha, x)$. Thus we have shown that the x-component of F_t, which we have called A, is the unique component of F_t whose closure is not a compact subset of U.

Let $\{W_i\}_{i \in I}$ be the collection of components of $U \backslash A$. Since $E \subseteq U \backslash A$ is connected, there is one such component W_{i_0} that contains E. Note that A is open in U, as U is locally connected and F_t is open [24, Theorem 25.3]. Thus $U \backslash A$ is closed in U, and so W_{i_0} is closed in U as well, as it is a component of a closed set. In fact, W_{i_0} is a compact subset of U, as follows. By assumption (i), the set $E_t \cup L_t$ is compact. As U is homeomorphic to the plane, there is a compact set $K \subseteq U$ such that $U \backslash K$ is connected and $E_t \cup L_t \subseteq K$. Then $U \backslash K$ must be contained in some component of F_t. Since $U \backslash K$ does not have compact closure in U, we must have $U \backslash K \subseteq A$. Thus $W_{i_0} \subseteq K$, which implies that W_{i_0} is compact.

Let

$$V = A \cup \{W_i\}_{i \neq i_0} = U \backslash W_{i_0}.$$

Since A is connected, each W_i is connected, and $\operatorname{cl}_U(A) \cap W_i \neq \emptyset$ for each i, we see that V is connected. Let $U \cup \{\infty\}$ denote the one-point compactification of U, which is homeomorphic to the sphere $\mathbb{S}^2$. Since W_{i_0} is a continuum, Lemma 4.14 implies that $V \cup \{\infty\}$, considered as a subset of $U \cup \{\infty\}$, is homeomorphic to the plane, and that $\operatorname{fr}_U(V)$ is connected.

We claim that $\operatorname{fr}_U(V) \subseteq L_t$. Since $V = U \backslash W_{i_0}$, we see that $\operatorname{fr}_U(V) = \operatorname{fr}_U(W_{i_0})$. As W_{i_0} is a component of the closed set $U \backslash A$ and U is locally connected,

$$\operatorname{fr}_U(V) \subseteq \operatorname{fr}_U A.$$

Since A is a component of F_t, it is relatively closed in F_t. This implies that $\operatorname{fr}_U(A) \cap F_t = \emptyset$. However, the continuity of the distance function implies that $\operatorname{fr}_U(A) \cap E_t = \emptyset$, and so $\operatorname{fr}_U(V) \subseteq L_t$.

Since $d(x, E) = r/8N$, Remark 4.7 provides a path from x to E inside $B(x, r/4) \subseteq U$. Since x is in A and $E \subseteq W_{i_0}$, this path must intersect $\operatorname{fr}_U(V)$ at some point

x'. Similarly there is a point $y' \in B(y, r/4) \cap \mathrm{fr}_U(V)$. Thus $\mathrm{fr}_U(V)$ is a closed and connected subset of L_t which connects $B(x, r/4)$ to $B(y, r/4)$. □

The measure estimate (4.5) actually allows us to prove more than just local quasiconvexity. We refer to [15] and [13] for the definition of the modulus of a curve family and the basic facts regarding modulus, Loewner spaces, and Poincaré inequalities.

Corollary 4.15. *Assume the hypotheses of Theorem 4.6. Then there is a constant $L \geq 0$ depending only on C, Λ, M, and N such that the following statement holds. Let $x, y \in B(z, R)$ and set $r = d(x, y)$. Then the 2-modulus of the path family connecting $B(x, r/4)$ to $B(y, r/4)$ is at least L.*

Proof. Let s_0 be as in the proof of Theorem 4.6, and set

$$\mathcal{G} = \{t \in [0, r/s_0] : \mathcal{H}^1(L_t) \leq 8C\omega N^2 s_0 r\}.$$

The measure estimate (4.5) shows that

$$\mathcal{H}^1(\mathcal{G}) \geq \frac{r}{2s_0}.$$

By the proof of Theorem 4.6 and the arc extraction discussed in Remark 3.15, for each $t \in \mathcal{G}$ we may find an arc $\gamma_t \subseteq L_t$ connecting $B(x, r/4)$ to $B(y, r/4)$. Thus it suffices to show that the 2-modulus of the family $\{\gamma_t : t \in \mathcal{G}\}$ is bounded away from zero.

We seek a lower bound for

$$\inf \int_X \rho^2 \, d\mathcal{H}^2,$$

where the infimum is taken over all Borel measurable functions $\rho\colon X \to [0, \infty]$ such that for all $t \in \mathcal{G}$

$$\int_{\gamma_t} \rho \, ds \geq 1. \tag{4.7}$$

Let ρ be such a function. We claim that the following weighted co-area inequality holds:

$$\int_{\mathcal{G}} \int_{\gamma_t} \rho^2 \, d\mathcal{H}^1 d\mathcal{H}^1 \leq \omega \int_X \rho^2 \, d\mathcal{H}^2. \tag{4.8}$$

If ρ^2 is the characteristic function of a $\mathcal{H}^2$-measurable subset $A \subseteq X$, then (4.8) follows from the usual co-area inequality given in Theorem 4.10 applied to the metric space $X \cap A$. Linearity of the integral then shows that (4.8) holds if ρ^2 is a simple function, and the standard limiting argument using the monotone convergence theorem shows that (4.8) holds in the desired generality.

Thus, applying (4.8) and Hölder's inequality,

$$\int_X \rho^2 \, d\mathcal{H}^2 \geq \omega^{-1} \int_{\mathcal{G}} \frac{1}{\mathcal{H}^1(\gamma(t))} \left(\int_{\gamma_t} \rho \, d\mathcal{H}^1 \right)^2 \, d\mathcal{H}^1.$$

Since γ_t is an arc,

$$\int_{\gamma_t} \rho \, d\mathcal{H}^1 = \int_{\gamma_t} \rho \, ds \geq 1,$$

and so by the definition of $\mathcal{G}$ and the measure estimate for $\mathcal{G}$, we have

$$\int_X \rho^2 \, d\mathcal{H}^2 \geq \omega^{-1} \int_{\mathcal{G}} \frac{1}{8C\omega N^2 s_0 r} \, d\mathcal{H}^1 \geq \frac{1}{16C\omega^2 N^2 s_0^2} =: L.$$

□

Remark 4.16. Arguing as in [5, Proposition 3.1], Corollary 4.15 implies that a locally Ahlfors 2-regular and $LLLC$ metric space (X,d) homeomorphic to a surface is locally 2-Loewner. As in [15, Section 5], this implies that (X,d) locally supports a weak $(1,2)$-Poincaré inequality. Using the techniques of [15, Theorem 5.9], Corollary 4.15 could probably be improved to show that (X,d) locally supports a weak $(1,1)$-Poincaré inequality. This would be the full local analog of Semmes' result [27, Theorem B.10] in dimension 2.

4.3. **Construction of quasicircles.** In this section, we construct chord-arc loops (and hence quasicircles) at specified scales under quite general hypotheses.

Theorem 4.17. *Let (X,d) be a metric space, and $z \in X$. Suppose that $U \subseteq X$ is a neighborhood of z homeomorphic to the plane $\mathbb{R}^2$, and that there are constants $\Lambda, D, L \geq 1$ and $Q \geq 0$ as well as a radius $R_0 > 0$ such that*

- (i) *$\bar{B}(z,R_0)$ is a compact subset of U,*
- (ii) *U has relative Assouad dimension at most Q with constant D,*
- (iii) *U is relatively Λ-linearly locally contractible,*
- (iv) *each pair of points $x,y \in B(z,R_0)$ may be connected by a path of length at most $Ld(x,y)$.*

Then there exist constants $\lambda, C_1, C_2 \geq 1$, depending only on Λ, D, Q and L, such that if $R \leq R_0/C_1$, there is a λ-chord-arc loop $\gamma\colon \mathbb{S}^1 \to B(z,R_0)$ with $\operatorname{ind}(\gamma,z) \neq 0$ and

$$\operatorname{im}(\gamma) \subseteq B(z,C_2R)\backslash B\left(z,\frac{R}{C_2}\right), \quad \textit{and} \quad \frac{R}{C_2} \leq \operatorname{diam}(\operatorname{im}(\gamma)) \leq C_2R. \tag{4.9}$$

The rest of this section will consist of the proof of this theorem, and so throughout we let (X,d), U, z, etc., be as in the hypotheses of Theorem 4.17. The construction has two steps. First, we create a polygon with a controlled number of vertices surrounding z at the correct scale. We then minimize a functional on loops surrounding z. The minimizer will be a chord-arc loop.

For $a,b \in X$, define

$$d'(a,b) = \inf \operatorname{length}_d(\gamma),$$

where the infimum is taken over all paths $\gamma\colon [0,1] \to X$ with $\gamma(0) = a$ and $\gamma(1) = b$. Assumption (iv) implies that d' is a metric on $B_d(z,R_0)$, and shows that if $a,b \in B_d(z,R_0)$, then

$$d(a,b) \leq d'(a,b) \leq Ld(a,b). \tag{4.10}$$

Note that the first inequality in (4.10) is valid for all points $a,b \in X$. Thus if $x \in B_d(z,R_0/2)$, and $0 < r \leq R_0/2$, we have

$$B_d\left(x,\frac{r}{L}\right) \subseteq B_{d'}(x,r) \subseteq B_d(x,r) \subseteq B_d(z,R_0). \tag{4.11}$$

By the compactness assumption (i), for any pair of points $x,y \in B_d(z,R_0/(2L))$, there is a (not necessarily unique) path in X whose d-length is $d'(x,y)$ [9, 2.5.19]. The following lemma shows that such a path is a geodesic in the d'-metric.

Lemma 4.18. *Let $\gamma\colon [0,1] \to X$ be a path such that*

$$\operatorname{length}_d(\gamma) = d'(\gamma(0),\gamma(1)).$$

Then $\operatorname{length}_{d'}(\gamma) = d'(\gamma(0),\gamma(1))$ *as well.*

Proof. We first claim that for all $s, t \in [0,1]$, we have

$$\text{length}_d(\gamma|_{[s,t]}) = d'(\gamma(s), \gamma(t)).$$

The definition of d' shows that

$$\text{length}_d(\gamma|_{[s,t]}) \geq d'(\gamma(s), \gamma(t)).$$

If this inequality is is strict, then there is a path β in X connecting $\gamma(s)$ to $\gamma(t)$ with

$$\text{length}_d(\beta) < \text{length}_d(\gamma|_{[s,t]}).$$

This implies that

$$\text{length}_d(\gamma|_{[0,s]} \cdot \beta \cdot \gamma|_{[t,1]}) < \text{length}_d(\gamma) = d'(\gamma(0), \gamma(1)),$$

a contradiction.

We now prove the lemma. Suppose that $0 = t_1 < \ldots < t_n = 1$ is a partition of $[0,1]$. Then by the claim,

$$\sum_{i=1}^{n-1} d'(\gamma(t_i), \gamma(t_{i+1})) = \sum_{i=1}^{n-1} \text{length}(\gamma|_{[t_i,t_{i+1}]}) = \text{length}(\gamma).$$

Since this is true for each partition, it is true for the supremum over all partitions. The lemma follows. □

We denote the image of a d'-geodesic connecting points x and y by $[x, y]$, following the conventions for geodesics laid out in Section 3.

Let

$$B_0 = \bar{B}_{d'}\left(z, \frac{R_0}{16\Lambda L}\right);$$

our construction will take place inside this set. Note that by (4.11) and assumption (i), the set B_0 is a compact subset of U, and hence

$$\frac{R_0}{16\Lambda L} \leq \text{diam}_{d'}(B_0) \leq \frac{R_0}{8\Lambda L}.$$

Lemma 4.19. *In the d'-metric, the set B_0 is relatively ΛL-linearly locally contractible, and has relative Assouad dimension at most Q with constant DL^Q.*

Proof. Let $x \in B_0$, and let $r \leq \text{diam}_{d'}(B_0) \leq R_0/8\Lambda L$. In particular, this implies that $x \in U$ and $r \leq \text{diam}_d(U)$.

We first show that $B_{d'}(x, r)$ contracts inside $B_{d'}(x, \Lambda L r)$. Since U is relatively Λ-linearly locally contractible, the ball $B_d(x, r)$ contracts inside $B_d(x, \Lambda r)$. Since $\Lambda L r \leq R_0/2$, we may apply (4.11) to see that $B_d(x, \Lambda r) \subseteq B_{d'}(x, \Lambda L r)$. Thus $B_d(x, r)$ contracts inside $B_{d'}(x, \Lambda L r)$. Since $B_{d'}(x, r) \subseteq B_d(x, r)$, this suffices.

Let $0 < \epsilon \leq 1/2$; we now show that the ball $B_{d'}(x, r)$ can be covered by a controlled number of d'-balls of radius ϵr. Since U has relative Assouad dimension at most Q with constant D, the ball $B_d(x, r)$ may be covered by at most $DL^Q \epsilon^{-Q}$ balls $\{B_i = B_d(x_i, \epsilon r/L)\}_{i \in I}$. We may assume that for each $i \in I$, $d(x_i, x) \leq r + \epsilon r/L$, for otherwise $B_i \cap B_d(x, r) = \emptyset$. Then for each $i \in I$,

$$d(x_i, z) \leq d(x_i, x) + d(x, z) \leq 2r + \frac{R_0}{16\Lambda L} \leq \frac{R_0}{2}.$$

Since $\epsilon r \leq R_0/2$, the inclusions (4.11) imply that $B_i \subseteq B_{d'}(x_i, \epsilon r)$ for each $i \in I$. Since $B_{d'}(x, r) \subseteq B_d(x, r)$, this completes the proof. □

The assumption of relative linear local contractibility implies that a loop which stays far away from a given point either has large diameter or has index zero with respect to that point.

Lemma 4.20. *Let $a > 0$, and let $0 < R \leq R_0/16a$. Suppose that $\alpha\colon \mathbb{S}^1 \to B_0$ is a continuous map with* $\operatorname{ind}(\alpha, z) \neq 0$ *and* $\operatorname{dist}_{d'}(z, \operatorname{im}(\alpha)) \geq aR$. *Then* $\operatorname{diam}_{d'}(\operatorname{im}\alpha) \geq aR/\Lambda L$.

Proof. Suppose that $\operatorname{diam}_{d'}(\operatorname{im}\alpha) < aR/\Lambda L$. Let $x \in \operatorname{im}\alpha$; then we have $\operatorname{im}\alpha \subseteq B_{d'}(x, aR/\Lambda L)$. The upper bound on R implies that $x \in B_0$ and $aR/\Lambda L \leq \operatorname{diam}_{d'}(B_0)$. By Lemma 4.19, B_0 is relatively ΛL-linearly locally contractible, and so α is homotopic to a point inside of $B_{d'}(x, aR)$. However, the assumption that $d'(z, x) \geq aR$ implies that the tracks of the homotopy do not meet z. This is a contradiction with the fact that $\operatorname{ind}(\alpha, z) \neq 0$. □

We now begin the construction of the polygon discussed above.

Lemma 4.21. *Let $0 < R \leq R_0/48\Lambda L$. Then there exists a path $\beta\colon \mathbb{S}^1 \to B_0$ and a constant C_0, depending only on D, Q, Λ, and L, such that the following statements hold:*

(i) $\operatorname{ind}(\beta, z) \neq 0$,
(ii) $\operatorname{im}(\beta) \subseteq B_{d'}(z, 3R) \backslash B_{d'}(z, R/2)$,
(iii) $\operatorname{length}_{d'}(\beta) \leq C_0 R$.

Proof. Set

$$\epsilon = \frac{R}{128\Lambda L(\Lambda L + 1)}.$$

The number ϵ will be roughly the distance between vertices of the polygon to be constructed. By the definition of the d'-metric, the ball $B_{d'}(z, R)$ is connected and has compact closure in U, so by Lemma 4.13 there is an embedding $\alpha\colon \mathbb{S}^1 \to B_{d'}(z, 2R) \subseteq B_0$ with $B_{d'}(z, R) \subseteq \operatorname{inside}(\operatorname{im}\alpha)$. Thus

$$\operatorname{im}(\alpha) \subseteq B_{d'}(z, 2R) \backslash B_{d'}(z, R) \quad \text{and} \quad \operatorname{ind}(\alpha, z) \neq 0. \tag{4.12}$$

This implies in particular, that $\operatorname{im}\alpha \subseteq B_0$. Let S be a maximal ϵ-separated set in $\operatorname{im}\alpha$, with respect to the d' metric. By Lemma 4.19, B_0 has relative Assouad dimension at most Q with constant DL^Q. Since $S \subseteq \operatorname{im}\alpha \subseteq B_{d'}(z, 2R)$, and $2R \leq \operatorname{diam}_{d'}(B_0)$, we see from the definition of ϵ that $\operatorname{card} S$ is bounded above by a number that depends only on D, Q, Λ and L.

By Proposition 4.12, we may find a finite and cyclically ordered set of points $\{\psi_1, \ldots, \psi_n\} \subseteq \mathbb{S}^1$ such that for $i = 1, \ldots, n$, $\operatorname{mod} n$,

$$\operatorname{diam}_{d'}(\alpha([\psi_i, \psi_{i+1}])) < \epsilon. \tag{4.13}$$

Note that we have no control over the size of n.

We inductively define a sequence of indices $\{i_k\} \subseteq \{1, \ldots, n\}$ and a sequence of points $\{z_k\} \subseteq S$ as follows. Let $i_i = 1$, and let $z_1 \in S$ be any point such that $d'(z_1, \alpha(\psi_{i_1})) < \epsilon$. Such a point exists since S was chosen to be maximal. Now suppose that $z_k \in S$ and $i_k \in \{1, \ldots, n\}$ have been chosen. Let i_{k+1} be the smallest index j greater than i_k such that $\alpha(\psi_j) \notin B_{d'}(z_k, \epsilon)$. See Figure 5. If no such index exists, the process stops. If i_{k+1} may be found, set z_{k+1} to be any point in S such that $d'(z_{k+1}, \alpha(\psi_{i_{k+1}})) < \epsilon$. Since $i_k < i_{k+1} \leq n$, this process stops after finitely many iterations. Let z_m and i_m be the final point and index produced.

The result of this process satisfies

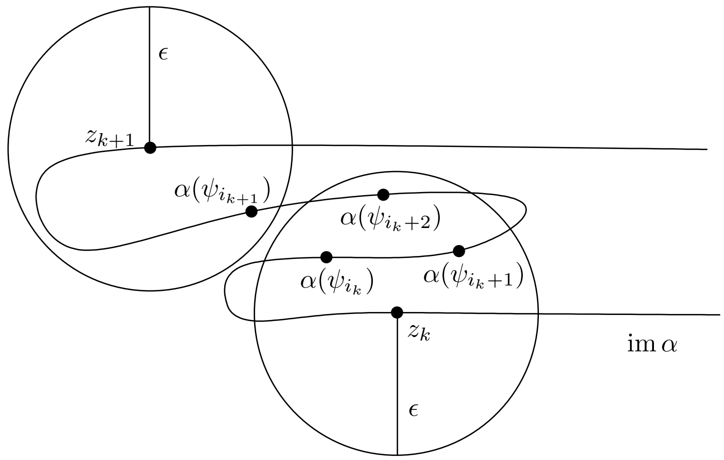

FIGURE 5. Choosing z_{k+1}

(a) $d'(\alpha(\psi_i), z_k) < \epsilon$ if $1 \le k < m$ and $i_k \le i < i_{k+1}$,
(b) $d'(\alpha(\psi_i), z_m) < \epsilon$ if $i_m \le i \le n$.

From (a) and (4.13), we see that if $1 \le k < m$,

$$d'(z_k, z_{k+1}) \le d'(z_k, \alpha(\psi_{i_{k+1}-1})) + d'(\alpha(\psi_{i_{k+1}-1}), \alpha(\psi_{i_{k+1}})) + d'(\alpha(\psi_{i_{k+1}}), z_{k+1}) < 3\epsilon,$$

and (b) shows that

$$d'(z_m, z_1) \le d'(z_m, \alpha(\psi_n)) + d'(\alpha(\psi_n), \alpha(\psi_1)) + d'(\alpha(\psi_1), z_1) < 3\epsilon.$$

These inequalities imply that

$$\text{length}_{d'}([z_k, z_{k+1}]) = \text{diam}_{d'}([z_k, z_{k+1}]) < 3\epsilon \tag{4.14}$$

for $k = 1, \dots, m$, $\bmod m$.

Furthermore, (a) and (b) together with (4.13) show that for $k = 1, \dots, m$, $\bmod m$,

$$\alpha([\psi_{i_k}, \psi_{i_{k+1}}]) \subseteq B_{d'}(z_k, 2\epsilon),$$

and so for $k = 1, \dots, m$, $\bmod m$,

$$\text{diam}_{d'}(\alpha([\psi_{i_k}, \psi_{i_{k+1}}])) \le 4\epsilon, \tag{4.15}$$

Rename $\psi_{i_k} = \theta_k$ for $k = 1, \dots, m$, and let $\beta\colon \mathbb{S}^1 \to X$ be defined by

$$\beta(\theta) = s^{[z_k, z_{k+1}]}_{[\theta_k, \theta_{k+1}]}(\theta), \;\; \theta \in [\theta_k, \theta_{k+1}].$$

We wish to show that α and β are homotopic away from z. To do so, we apply Lemma 4.9 to α and β, using $\delta = 4\epsilon$. We now verify the hypotheses of Lemma 4.9. Recall that by Lemma 4.19, B_0 is relatively ΛL-linearly locally contractible in the d'-metric, and that $R_0/(16\Lambda L) \le \text{diam}_{d'} B_0$. If $x \in B_0$ and $0 < r \le 8\epsilon(\Lambda L + 1)$, the definition of ϵ and the assumption that $R \le R_0$ show that $r \le \text{diam}_{d'} B_0$. Thus the ball $B_{d'}(x, r)$ contracts inside $B_{d'}(x, \Lambda L r)$. The inequalities (a), (b), (4.14), and (4.15) show that the remaining hypotheses of Lemma 4.9 are fulfilled, showing that β is homotopic to α inside the $8\Lambda L(\Lambda L + 1)\epsilon$-neighborhood of α (in the d'-metric). Since $8\Lambda L(\Lambda L + 1)\epsilon = R/16$ and $\text{dist}_{d'}(z, \operatorname{im}\alpha) \ge R$, the tracks of the homotopy do not meet z. Conclusions (i) and (ii) now follow from (4.12).

It could be the case that $z_k = z_l$ for $k \neq l$. As a result, we may not conclude that $m \leq \operatorname{card} S$. However, we may decompose $z_1, \ldots, z_m$ in to a finite collection of cycles where no z_i is repeated. The map β can then be considered as the concatenation of the restrictions to corresponding parameter segments. Since $\operatorname{ind}(\beta, z) \neq 0$, at least one such restriction must also have non-zero index. As the resulting loop is a subset of $\operatorname{im}\beta$, conclusions (i) and (ii) persist. Repeating this procedure finitely many times, we may assume without loss of generality that $z_1, \ldots, z_m \subseteq S$ are distinct points. As $\operatorname{card} S$ depends only on D, Q, Λ, and L, the conclusion (iii) now follows from (4.14). Note that by conclusion (ii) and the assumption that $R \leq R_0/48\Lambda L$, the image of β is contained in B_0. □

To complete the proof of Theorem 4.17, we need the following technical fact regarding lower semi-continuity of path integrals.

Lemma 4.22. *Let $\rho : X \to [0, \infty)$ be a lower semi-continuous function on a metric space (X, d), and suppose that $\{\gamma_n\}_{n \in \mathbb{N}}$ is a sequence of loops in X of uniformly bounded length. If γ_n converges uniformly to a loop γ in X, then*

$$\int_\gamma \rho \, ds \leq \liminf_{n \to \infty} \int_{\gamma_n} \rho \, ds.$$

Proof. The proof of this fact is given in the last three paragraphs of the proof of [15, Prop. 2.17]. The key fact is that a 1-Lipschitz function $s : I \to \mathbb{R}$, where I is any interval, is differentiable almost everywhere and satisfies $s'(t) \leq 1$ almost everywhere. □

Proof of Theorem 4.17. Let C_0 be the constant provided by Lemma 4.21, and set

$$C_1 = 320 C_0 (2\Lambda L + 1)\Lambda L.$$

Fix $0 < R \leq R_0/C_1$.

Define the continuous function $\rho\colon B_0 \backslash \{z\} \to [0, \infty)$ by

$$\rho(x) = \left(\frac{R}{d'(x, z)}\right)^2 + 1,$$

and for any rectifiable loop $\gamma\colon \mathbb{S}^1 \to B_0 \backslash \{z\}$, define the functional

$$\sigma(\gamma) = \int_\gamma \rho \, ds.$$

The functional σ balances the length of a loop against its distance to z.

If β is the loop given by Lemma 4.21, we have

$$0 < \sigma(\beta) \leq 5 \operatorname{length}_{d'}(\beta) \leq 5 C_0 R.$$

For $n \in \mathbb{N}$, we may find rectifiable loops $\gamma_n\colon \mathbb{S}^1 \to B_0 \backslash \{z\}$, such that $\operatorname{ind}(\gamma_n, z) \neq 0$ and

$$\lim_{n \to \infty} \sigma(\gamma_n) = \inf \sigma(\gamma),$$

where the infimum is taken over all rectifiable loops γ in $B_0 \backslash \{z\}$ with $\operatorname{ind}(\gamma, z) \neq 0$. Without loss of generality, we may assume that $\sigma(\gamma_n) < 2\sigma(\beta) \leq 10 C_0 R$, for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$. Our first task is to show that the loop γ_n lies in a controlled annulus around z. Define

$$d_n = \min\{d'(z, \gamma_n(\theta)) : \theta \in \mathbb{S}^1\} \quad \text{and} \quad D_n = \max\{d'(z, \gamma_n(\theta)) : \theta \in \mathbb{S}^1\},$$

and set $l_n = \text{length}_{d'}(\gamma_n)$. We have that

$$l_n \le \sigma(\gamma_n) < 10C_0R. \tag{4.16}$$

Suppose that $d_n \ge 2\Lambda L l_n$. For $x \in \text{im}(\gamma_n) \subseteq B_0$, we have that

$$\text{im}(\gamma_n) \subseteq B_{d'}(x, 2l_n).$$

Recall that B_0 is relatively ΛL-linearly locally contractible by Lemma 4.19. From (4.16) and the definition of C_1, we see that $2l_n \le \text{diam}_{d'} B_0$. Thus γ_n is homotopic to a point inside of $B(x, 2\Lambda L l_n) \subseteq U$. However, $d'(x,z) \ge d_n \ge 2\Lambda L l_n$. This is a contradiction with the assumption that $\text{ind}(\gamma_n, z) \ne 0$. Thus

$$d_n < 2\Lambda L l_n. \tag{4.17}$$

The estimates (4.16) and (4.17) now imply that

$$D_n < d_n + l_n < 10C_0(2\Lambda L + 1)R. \tag{4.18}$$

We now derive a lower bound for d_n. We do so in two cases. First assume that $D_n \le 4d_n$. Using (4.17), we have

$$10C_0R \ge \sigma(\gamma_n) \ge \int_{\gamma_n} \left(\frac{R}{4d_n}\right)^2 ds = \left(\frac{R}{4d_n}\right)^2 l_n \ge \frac{R^2}{32\Lambda L d_n}.$$

Thus

$$d_n \ge \frac{R}{320C_0\Lambda L}. \tag{4.19}$$

Now assume that $D_n > 4d_n$. It follows from the triangle inequality that

$$10C_0R \ge \sigma(\gamma_n) \ge \int_{d_n}^{D_n} \left(\frac{R}{t}\right)^2 dt = R^2\left(\frac{1}{d_n} - \frac{1}{D_n}\right) \ge \frac{3R^2}{4d_n},$$

yielding $d_n \ge 3R/40C_0$. In either case, (4.19) holds.

The compactness of $\mathbb{S}^1$ and the length bound (4.16) imply that the family $\{\gamma_n\}_{n\in\mathbb{N}}$ is equicontinuous. By the Arzela-Ascoli Theorem, after passing to a subsequence, the loops γ_n converge uniformly to a loop $\gamma_0 \colon \mathbb{S}^1 \to B_0$ such that the following hold:

$$\text{length}(\gamma_0) \le 10C_0R, \tag{4.20}$$

$$d_0 := \min\{d'(\gamma_0(\theta), z) : \theta \in \mathbb{S}^1\} \ge \frac{R}{320C_0\Lambda L}, \tag{4.21}$$

$$D_0 := \max\{d'(\gamma_0(\theta), z) : \theta \in \mathbb{S}^1\} \le 10C_0(2\Lambda L + 1)R. \tag{4.22}$$

Using the fact that $\gamma_n \to \gamma_0$ uniformly and that d' is a "geodesic" metric on B_0, we see from (4.19) and (4.21) that for sufficiently large $n \in \mathbb{N}$, the loop γ_n is homotopic to γ_0 in $U\backslash\{z\}$. Thus $\text{ind}(\gamma_0, z) \ne 0$.

The estimate (4.21) shows that we may apply Lemma 4.20 to γ_0 with $a = (320C_0\Lambda L)^{-1}$, concluding that

$$\text{diam}_{d'}(\text{im}(\gamma_0)) \ge \frac{R}{320C_0(\Lambda L)^2}.$$

From these facts and the comparability of the metrics d and d' given in (4.10), we conclude that there is a constant C_2, depending only on Λ, D, Q and L, such that γ_0 satisfies the conditions in (4.9).

It remains to show that γ_0 is a chord-arc circle with an appropriate constant. Since $\gamma_0 \subseteq B_0$, the comparability of d' and d stated in (4.10) shows that it suffices to check the chord-arc condition in the d' metric. Let $\phi, \psi \in \mathbb{S}^1$, and let J_1 and J_2 denote the subarcs of $\mathbb{S}^1$ whose union is $\mathbb{S}^1$ and whose intersection is $\{\phi, \psi\}$. We will show that

$$\min\{\text{length}_{d'}(\gamma_0|_{J_1}), \text{length}_{d'}(\gamma_0|_{J_2})\} \leq (6400C_0^2\Lambda L(2\Lambda L+1))^2 d'(\gamma_0(\phi), \gamma_0(\psi)). \tag{4.23}$$

We first assume that

$$d'(\gamma_0(\phi), \gamma_0(\psi)) \leq \frac{R}{640C_0\Lambda L}. \tag{4.24}$$

By (4.22) and the definition of C_1, the geodesic segment $[\gamma_0(\phi), \gamma_0(\psi)]$ is contained in B_0. By (4.21), it does not meet z. By the additivity of index under concatenation, we may assume without loss of generality that the loop

$$\widetilde{\gamma}_0(\theta) = \begin{cases} \gamma_0(\theta) & \theta \in J_1 \\ s_{J_2}^{[\gamma_0(\phi), \gamma_0(\psi)]}(\theta) & \theta \in J_2 \end{cases}$$

satisfies $\text{ind}(\widetilde{\gamma}_0, z) \neq 0$. See Figure 6. Lemma 4.22 now implies that $\sigma(\gamma_0) \leq \sigma(\widetilde{\gamma}_0)$.

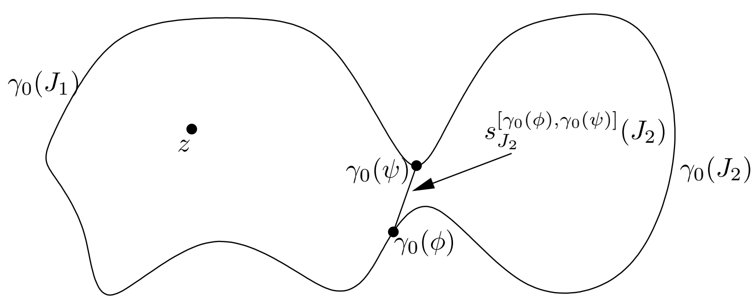

FIGURE 6. A shortcut

Note that for $\theta \in J_2$, (4.21) and (4.24) imply

$$d'(\widetilde{\gamma}_0(\theta), z) \geq \frac{R}{640C_0\Lambda L}.$$

Using this, (4.22), and the fact that γ_0 and $\widetilde{\gamma}_0$ agree on J_1, we see that

$$\begin{aligned} 0 \geq \sigma(\gamma_0) - \sigma(\widetilde{\gamma}_0) \geq &\left((10C_0(2\Lambda L+1))^{-2}+1\right)\text{length}_{d'}(\gamma_0|_{J_2}) \\ &- \left((640C_0\Lambda L)^2+1\right)\text{length}_{d'}(\widetilde{\gamma}_0|_{J_2}). \end{aligned}$$

Recalling that $\text{length}_{d'}(\widetilde{\gamma}_0|_{J_2}) = d'(\gamma_0(\phi), \gamma_0(\psi))$, this yields (4.23).

We now assume

$$d'(\gamma_0(\phi), \gamma_0(\psi)) \geq \frac{R}{640C_0\Lambda L}.$$

By (4.20),

$$\min\{\text{length}(\gamma_0|_{J_1}), \text{length}(\gamma_0|_{J_2})\} \leq \text{length}(\gamma_0) \leq (6400C_0^2\Lambda L)d'(\gamma_0(\phi), \gamma_0(\psi)).$$

This verifies (4.23), showing that γ_0 is a chord-arc loop with an appropriate constant, and completes the proof. □

4.4. **Porosity of quasicircles.** We now show, in particular, that a quasicircle that is the metric boundary of a metric disk is porous in the completed space. For quasicircles in the plane, this result is well known. We will use this property to get around the fact that subsets of an Ahlfors regular space need not be Ahlfors regular.

Definition 4.23. A subset Z of a metric space (X, d) is C-porous, $C \geq 1$, if for every $z \in Z$ and $0 < r \leq \operatorname{diam}(X)$, there exists a point $x \in X$ such that

$$B\left(x, \frac{r}{C}\right) \subseteq B(z, r) \backslash Z.$$

Theorem 4.24. *Let $\lambda \geq 1$, and suppose that (X, d) is a metric space homeomorphic to the plane such that $\bar{X}$ is compact and λ-LLC, and ∂X is a λ-LLC Jordan curve. Then ∂X is porous in $\bar{X}$ with constant depending only on λ.*

We will need a version of Janiszewski's separation theorem. We show how the variant can be derived from the original, a proof of which may be found in [25, V.9]. A subset A of a topological space X is said to separate points $u, v \in X$ if u and v are in different components of $X \backslash A$.

Theorem 4.25 (Janiszewski's Theorem)**.** *Let A and B be disjoint closed subsets of $\mathbb{R}^2$. If $u, v \in \mathbb{R}^2$ are such that neither A nor B separates u and v, then $A \cup B$ does not separate u and v.*

Theorem 4.26 (Janiszewski's Theorem in $\bar{\mathbb{D}}^2$)**.** *Let $A, B \subseteq \bar{\mathbb{D}}^2$ be disjoint continua. If $u, v \in \bar{\mathbb{D}}^2$ are such that neither A nor B separates u from v, then $A \cup B$ does not separate u from v.*

Proof. Since A does not separate u from v, we may find a path $\gamma \colon [0, 1] \to \bar{\mathbb{D}}^2$ such that $\gamma(0) = u$ and $\gamma(1) = v$ and satisfying $A \cap \operatorname{im} \gamma = \emptyset$. Similarly, we have a path $\beta \colon [0, 1] \to \bar{\mathbb{D}}^2$ such that $\beta(0) = u$, $\beta(1) = v$, and $B \cap \operatorname{im} \beta = \emptyset$. Since A and B are compact, we may find $0 < \epsilon < 1$ such that

$$\{(1-\delta)u : 0 < \delta \leq \epsilon\} \cap (A \cup B) = \emptyset = \{(1-\delta)v : 0 < \delta \leq \epsilon\} \cap (A \cup B). \tag{4.25}$$

Furthermore, since $\operatorname{im} \gamma$ and $\operatorname{im} \beta$ are also compact, we may find $0 < \epsilon' \leq \epsilon$ so that the paths $\gamma' \colon [0, 1] \to \mathbb{D}^2$ and $\beta' \colon [0, 1] \to \mathbb{D}^2$ defined by

$$\gamma'(t) = (1-\epsilon')\gamma(t) \quad \text{and} \quad \beta'(t) = (1-\epsilon')\beta(t)$$

do not intersect $(A \cup B)$. Let $u' = (1-\epsilon')u$ and $v' = (1-\epsilon')v$. Note that $u', v' \in \mathbb{D}^2$. Then γ' connects u' to v' without intersecting A, and β' connects u' to v' outside of B. Since $\mathbb{D}^2$ is homeomorphic to $\mathbb{R}^2$, Theorem 4.25 provides a continuum $E \subseteq \mathbb{D}^2$ containing u' and v' such that $E \cap (A \cup B) = \emptyset$. Let $\gamma_u \colon [0, \epsilon'] \to \bar{\mathbb{D}}^2$ be defined by $\gamma_u(t) = (1-t)u$, and similarly define γ_v. Then by (4.25), $\operatorname{im}(\gamma_u) \cup E \cup \operatorname{im}(\gamma_v)$ is a continuum connecting u to v which does not intersect $A \cup B$, as desired. □

Proof of Theorem 4.24. We may assume without loss of generality that the boundary ∂X satisfies the so-called three point condition given by (3.4) with constant λ. Let $z \in \partial X$, and let $0 \leq r \leq \operatorname{diam}(X)$. We consider three cases.

Case 1: $0 \leq r < \operatorname{diam}(\partial X)/4\lambda$. In this case, we may find a point $w \in \partial X$ such that $d(z, w) \geq 2\lambda r$. We may also find points $u, v \in \partial X$ such that $\{z, v, w, u\}$ is cyclically ordered on ∂X, $d(z, u) = r/4\lambda = d(z, v)$, and if $J(z)$ is the component of $\partial X \backslash \{u, v\}$ which contains z, then $J(z) \subseteq B(z, r/4\lambda)$.

Let A be the component of $\bar{B}(z, r/8\lambda^2)$ containing z, and let B be the component of $\bar{X}\backslash B(z, r/2\lambda)$ containing w. Since $\bar{X}$ is λ-LLC, we see that

$$B\left(z, \frac{r}{8\lambda^3}\right) \subseteq A, \quad \text{and} \quad \bar{X}\backslash B(z, r/2) \subseteq B. \tag{4.26}$$

As components of compact sets, A and B are continua, and are disjoint by definition. By definition, $\{u\}\cup J(z)\cup\{v\}$ connects u to v inside $\bar{B}(z, r/4\lambda) \subseteq \bar{X}\backslash B$. Furthermore, the LLC_2 condition shows that u and v may also be connected in $\bar{X}\backslash B(z, r/4\lambda^2) \subseteq \bar{X}\backslash A$. Thus by Theorem 4.26, there is a continuum $\alpha \subseteq \bar{X}\backslash(A\cup B)$ which contains both u and v. By (4.26), this implies that

$$\alpha \subseteq \left(\bar{X}\backslash B\left(z, \frac{r}{8\lambda^3}\right)\right) \cap B(z, r/2). \tag{4.27}$$

See Figure 7.

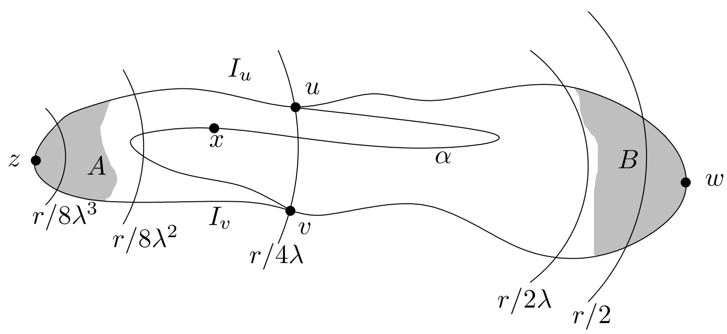

FIGURE 7. After applying Janiszewski's Theorem

Let $J(u)$ be the component of $\partial X\backslash\{z, w\}$ containing u, and set $I(u) = \{z\} \cup J(u) \cup \{w\}$. Define $I(v)$ similarly. We claim that $\operatorname{dist}(v, I(u)) \geq r/8\lambda^2$. If this is not the case, the three-point condition implies that either z or w is within distance $r/8\lambda$ of v, which is not the case. Thus by the connectedness of α, we may find a point $x \in \alpha$ such that

$$\operatorname{dist}(x, I(u)) = \frac{r}{32\lambda^4}.$$

Suppose there exists a point $y \in I(v)$ such that $d(x, y) < r/32\lambda^4$. Then $\operatorname{dist}(y, I(u)) \leq r/16\lambda^4$. Since ∂X satisfies the λ three-point condition 3.4, this implies that either z or w is contained in $B(y, r/16\lambda^3)$. However,

$$B\left(y, \frac{r}{16\lambda^3}\right) \subseteq B\left(x, \frac{r}{16\lambda^3} + \frac{r}{32\lambda^4}\right) \subseteq B\left(x, \frac{r}{8\lambda^3}\right),$$

and that $x \in \alpha$. By (4.27) and the fact that $d(z, w) \geq 2\lambda r$, this yields a contradiction. Thus we see that $\operatorname{dist}(x, I(v)) \geq r/32\lambda^4$. This along with (4.27) shows that

$$B\left(x, \frac{r}{32\lambda^4}\right) \subseteq B(z, r)\backslash\partial X.$$

Case 2: $8\operatorname{diam}(\partial X) \leq r \leq \operatorname{diam} X$. We may find a point $x \in \bar{X}$ such that $d(x, z) = r/4$. Since $z \in \partial X$, and $\operatorname{diam} \partial X \leq r/8$, we see that

$$\operatorname{dist}(x, \partial X) \geq d(x, z) - \operatorname{diam} \partial X \geq \frac{r}{8}.$$

Thus $B(x, r/8) \subseteq B(z, r)\backslash\partial X$.

Case 3: $\operatorname{diam}(\partial X)/4\lambda \le r \le 8\operatorname{diam}(\partial X)$. We have that

$$\frac{r}{64\lambda} < \frac{\operatorname{diam}(\partial X)}{4\lambda},$$

and so by Case 1, there is a point $x \in X$ such that

$$B\left(x, \frac{r}{2048\lambda^5}\right) \subseteq B\left(z, \frac{r}{64\lambda}\right)\backslash\partial X \subseteq B(z, r)\backslash\partial X.$$

These cases show that ∂X is $2048\lambda^5$-porous in $\bar{X}$. □

4.5. **The proof of Theorem 4.1.** In this subsection, we collect the results proven thus far and complete the proof of Theorem 4.1. We begin with a theorem that identifies planar sets on a surface, for which we were unable to find a reference.

Proposition 4.27. *Let $\mathcal{S}$ be a surface, and let $U \subseteq \mathcal{S}$ be a connected, non-empty, and open subset with compact closure such that $\mathcal{S}\backslash U$ is non-empty and connected, and the homomorphism $i_*\colon \pi_1(U) \to \pi_1(\mathcal{S})$ induced by the inclusion $i\colon U \to \mathcal{S}$ is trivial. Then U is homeomorphic to the plane.*

Proof. It suffices to show that U is simply connected, as follows. If U is simply connected, then it is orientable [28, 6.2.10]. Every connected, orientable surface has a Riemann surface structure [1, II.1.5E]. As $\mathcal{S}$ is connected and $\emptyset \neq U \neq \mathcal{S}$, the set U cannot be compact. The classical Uniformization Theorem now implies that U is homeomorphic to the plane.

We first consider the case that $\mathcal{S}$ is not compact. Suppose that U is not simply connected. Any continuous loop $\gamma\colon \mathbb{S}^1 \to \mathcal{S}$ is homotopic to a loop with only transversal self-intersections [12, Ch. 2 Sec. 3]. Thus we may find a loop in U with only finitely many self-intersections which represents a non-trivial homotopy class. By decomposing this loop, we may find a Jordan curve J in U which represents a non-trivial homotopy class.

We now claim that there is an embedding $h\colon \mathbb{D}^2 \to \mathcal{S}$ such that the topological boundary of $h(\mathbb{D}^2)$ is J. By the Uniformization Theorem, the universal cover $\widetilde{\mathcal{S}}$ of $\mathcal{S}$ is homeomorphic to the plane or the sphere. Since J is null-homotopic, the pre-image of J under the universal covering map is a collection of disjoint Jordan curves. By Schoenflies' theorem, each such curve is the boundary of an embedded disk D in $\bar{\mathcal{S}}$. The group of deck transformations acts without fixed points and moves each pre-image of J off of itself. Again by Schoenflies' theorem, we see that if g is a deck transformation with $g(D) \cap D \neq \emptyset$, then either $g(D) \subseteq D$ or $g^{-1}(D) \subseteq D$. In either case, the Brouwer fixed-point theorem yields a contradiction. Thus the covering projection restricted to D is a homeomorphism, proving the claim.

As J represents a non-trivial homotopy class in U, we see that $h(\mathbb{D}^2) \cap (\mathcal{S}\backslash U) \neq \emptyset$. Since $J \subseteq U$, the intersection $h(\mathbb{D}^2) \cap (\mathcal{S}\backslash U)$ is a relatively open and closed subset of $\mathcal{S}\backslash U$, and hence it is all of $\mathcal{S}\backslash U$. Since U has compact closure, this implies that $\mathcal{S}$ is compact, a contradiction.

We now assume that $\mathcal{S}$ is compact. We will make a homological argument; all homology groups will be singular and have coefficients in $\mathbb{Z}_2$. The reason for this is that any manifold has a unique orientation over $\mathbb{Z}_2$ [28, 6.2.9], and we will eventually use a duality theorem that requires an orientation. The long exact sequence for homology of the pair $(\mathcal{S}, U)$ includes

$$\ldots \to H_2(U) \to H_2(\mathcal{S}) \to H_2(\mathcal{S}, U) \to H_1(U) \to H_1(\mathcal{S}) \to \ldots.$$

By a version of Alexander duality [28, 6.2.17 and 6.9.9], there is a natural isomorphism

$$H_2(\mathcal{S}, U) \cong \check{H}^0(\mathcal{S} \backslash U),$$

where $\check{H}^*$ denotes Cěch cohomology with coefficients in $\mathbb{Z}_2$. Since $\mathcal{S} \backslash U$ is connected, this implies that $H_2(\mathcal{S}, U) \cong \mathbb{Z}_2$. As U is open and non-compact, $H_2(U) = 0$. By exactness,

$$H_2(\mathcal{S}) \to H_2(\mathcal{S}, U) \tag{4.28}$$

is injective. On the other hand, $\mathcal{S}$ is orientable over $\mathbb{Z}_2$ and compact, and so $H_2(\mathcal{S}) \cong \mathbb{Z}_2$. Thus the homomorphism (4.28) is surjective as well. Exactness yields that

$$H_2(\mathcal{S}, U) \to H_1(U) \tag{4.29}$$

is trivial.

Since $i_* \colon \pi_1(U) \to \pi_1(\mathcal{S})$ is trivial, the homomorphism

$$H_1(U) \to H_1(\mathcal{S})$$

is trivial as well. Exactness implies that (4.29) is surjective, and hence that $H_1(U)$ vanishes. Since U is open and non-compact, $\pi_1(U)$ is a free group [1, I.44]. Thus if $\pi_1(U)$ is non-trivial, then $H_1(U; \mathbb{Z})$ is a free abelian group of positive rank. By the Universal Coefficient Theorem for homology [8, Theorem 15.4(a)], $H_1(U)$ contains a subgroup isomorphic to $H_1(U; \mathbb{Z}) \otimes \mathbb{Z}_2$, which is non-trivial. This is a contradiction, and so we conclude that $\pi_1(U)$ is trivial, as desired. □

We now show that on an *LLLC* metric surface, the radius associated with the contractibility condition determines the size of planar sets in the surface.

Proposition 4.28. *Let (X, d) be an LLLC metric space homeomorphic to a surface, let $K \subseteq X$ a compact set, and let R_K and Λ_K be the radius and constant associated to K by the LLLC-condition. If $z \in K$ and $0 < R \leq R_K/8\Lambda_K$ is such that $B(z, 4\Lambda_K R) \subseteq K$, then there exists a neighborhood U of z which is homeomorphic to the plane and satisfies*

$$B\left(z, \frac{R}{2\Lambda_K}\right) \subseteq U \subseteq B(z, R). \tag{4.30}$$

Proof. Let (X, d), K, R_K, and Λ_K be as in the statement, and fix $z \in K$ and $0 < R \leq R_K/8\Lambda_K$.

If $X \backslash B(z, R)$ is empty, then $X = K$ is compact. Moreover, the definition of R shows that in this case, X is contractible. This is a contradiction, as there are no compact contractible surfaces. Thus we may assume $X \backslash B(z, R) \neq \emptyset$.

We first claim that $X \backslash B(z, R)$ is contained in a single component of $X \backslash B(z, R/(2\Lambda_K))$. By assumption, the ball $B(z, 4\Lambda_K R)$ is contained in K and its diameter is no greater than R_K. Thus by Proposition 3.14, it satisfies the first relative *LLC* condition with constant Λ_K. Let V be the connected component of $B(z, 4\Lambda_K R)$ that contains the point z. Then $B(z, 2R)$ is compactly contained in V. As X is locally connected, V is open [24, Theorem 25.3]. Thus by Proposition 3.14, V is relatively $2\Lambda_K$-*LLC*.

Suppose that $X\backslash B(z,R)$ intersects distinct components A and B of $X\backslash B(z,R/(2\Lambda_K))$. As X is connected and locally connected, both A and B must also intersect $\bar{B}(z,R/(2\Lambda_K))$. By continuity and connectedness, we may find points $a \in A$ and $b \in B$ with $d(a,z) = R = d(b,z)$. However, the second relative LLC condition implies that a and b may be connected in $V\backslash B(z,R/2\Lambda_K)$. This is a contradiction, proving the claim.

Let W be the component of $X\backslash B(z,R/2\Lambda_K)$ that contains $X\backslash B(z,R)$, and set $U = X\backslash W$. By definition, the inclusions (4.30) hold. Moreover, U is an open and non-compact subset of X with connected complement. The $LLLC$ condition implies that $U \subseteq B(z,R)$ contracts inside $B(z,\Lambda_K R)$. Thus the homomorphism $i_*\colon \pi_1(U) \to \pi_1(X)$ induced by the inclusion $i\colon U \to X$ is trivial. As a subset of K, the set U has compact closure in X. Proposition 4.27 now shows that U is homeomorphic to the plane. □

Proof of Theorem 4.1. We recall the set up. The space (X,d) is an $LLLC$ and locally Ahlfors 2-regular metric space homeomorphic to a surface. The set K is a compact subset of X such that if $x \in K$ and $0 < r \leq R_K$, then the ball $B(x,r)$ contracts inside $B(x,\Lambda_K r)$, and

$$\frac{r^2}{C_K} \leq \mathcal{H}^2(B(x,r)) \leq C_K r^2.$$

We let z be an interior point of K, and set

$$R_0 = \min\{\max\{R \geq 0 : \bar{B}(z,R) \subseteq K\}, R_K\} > 0.$$

If $X = B(z,R_0)$, then X is compact and contractible, a contradiction. Thus we may assume that $X\backslash B(z,R_0) \neq \emptyset$.

Consider the ball $B_0 = B(z,R_0/8)$. The definition of R_0 implies that $\operatorname{diam}(B_0) \leq R_K/4$, and that the $2\operatorname{diam}(B_0)$-neighborhood of B_0 is contained in K. This easily implies that B_0 is relatively Ahlfors 2-regular and relatively linearly locally contractible with constants C_K and Λ_K respectively. In addition, Proposition 3.6 implies that B_0 has relative Assouad dimension 2 with constant $64C_K^2$. Similarly, Proposition 3.14 shows that B_0 satisfies the first relative LLC condition with constant Λ_K.

If $x,y \in B(z,R_0/32) \subseteq B_0$, then $2d(x,y) \leq \operatorname{diam}(B_0)$. Since $x,y \in B(x,2d(x,y))$, the first relative LLC condition shows that there is a continuum connecting x to y of diameter no greater than $4\Lambda_K d(x,y)$. Thus $B(z,R_0/32)$ is of $4\Lambda_K$-bounded turning in X. Furthermore, as a subset of B_0, the ball $B(z,R_0/32)$ has relative Assouad dimension at most 2 with constant depending only on C_K. By Proposition 4.3, there are constants $M,N,c \geq 1$ depending only on C_K and Λ_K, such that for each pair of points $x,y \in B(z,R_0/(32c))$ and each $0 < \epsilon < d(x,y)$, there is an (ϵ,M)-quasiarc connecting x to y inside $B(x,Nd(x,y))$.

By the definition of R_0, we have

$$\frac{R_0}{32\Lambda_K c} \leq \frac{R_K}{8\Lambda_K} \quad \text{and} \quad B\left(z,\frac{4\Lambda_K R_0}{32\Lambda_K c}\right) \subseteq K.$$

Thus by Proposition 4.28, there is a neighborhood U of z homeomorphic to $\mathbb{R}^2$ such that

$$B\left(z,\frac{R_0}{64\Lambda_K^2 c}\right) \subseteq U \subseteq B\left(z,\frac{R_0}{32\Lambda_K c}\right). \tag{4.31}$$

In particular, $U \subseteq B_0$, and so U is also relatively Ahlfors 2-regular and relatively linearly locally contractible with constants C_K and Λ_K respectively. Furthermore, if $x, y \in U$, and $0 < \epsilon < d(x,y)$, there is an (ϵ, M)-quasiarc connecting x to y inside $B(x, Nd(x,y))$. It is easy to verify that

$$\bar{B}\left(z, \frac{4NR_0}{512\Lambda_K^2 Nc}\right)$$

is a compact subset of U. Thus by Theorem 4.6 there is a constant $L \geq 1$, depending only on C_K and Λ_K, such that each pair of points

$$x, y \in B\left(z, \frac{R_0}{512\Lambda_K^2 Nc}\right)$$

may be connected by a path of length no more than $Ld(x,y)$.

As a subset of B_0, the set U has relative Assouad dimension at most 2 with constant depending only on C_K. Theorem 4.17 shows that there are constants $\lambda, C_1, C_2 \geq 1$ depending only on C_K and Λ_K such that if $0 < R \leq R_0/C_1$, then there is a λ-chord-arc loop γ in U such that $\operatorname{ind}(\gamma, z) \neq 0$,

$$\operatorname{im}(\gamma) \subseteq B(z, C_2R) \backslash B\left(z, \frac{R}{C_2}\right), \quad \text{and} \quad \frac{R}{C_2} \leq \operatorname{diam}(\operatorname{im}(\gamma)) \leq C_2R. \tag{4.32}$$

Let $A_1 = 192C_2c\Lambda_K^2(4\Lambda_K + 2)$, and fix $0 < R \leq R_0/A_1$. Let γ be as described above, and set Ω to be the inside of $\operatorname{im}\gamma$. Then $\bar{\Omega} = \Omega \cup \operatorname{im}(\gamma)$. By definition, $\bar{\Omega}$ is a compact subset of U. We first claim that

$$\bar{\Omega} \subseteq B(z, C_2(4\Lambda_K + 2)R). \tag{4.33}$$

Suppose that $x \in \bar{\Omega}$ but $d(z,x) \geq C_2(4\Lambda_K + 2)R$. Then by (4.32)

$$\operatorname{dist}(x, \gamma) \geq d(z,x) - \operatorname{dist}(z, \gamma) - \operatorname{diam}(\operatorname{im}(\gamma)) \geq 4C_2\Lambda_K R > 0. \tag{4.34}$$

This shows that $x \in \Omega$, and so $\operatorname{ind}(\gamma, x) \neq 0$. However, U is relatively Λ_K-linearly locally contractible, and so γ is homotopic to a point with homotopy tracks inside the $2\Lambda_K \operatorname{diam}(\gamma)$ neighborhood of itself. By (4.32) and (4.34), these tracks do not hit x. This is a contradiction, proving (4.33).

We now claim that

$$B\left(z, \frac{R}{2\Lambda_K C_2}\right) \subseteq \Omega. \tag{4.35}$$

Let $C(z)$ be the z-component of $B(z, R/C_2)$. As $z \in \Omega$, (4.32) implies that $C(z) \subseteq \Omega$. By Proposition 3.14, the set U is relatively $2\Lambda_K$-LLC. This implies that $B(z, R/(2C_2\Lambda_K)) \subseteq C(z)$, establishing the claim. From (4.35) and (4.33), we see that Ω satisfies conclusion (i) of Theorem 4.1 with $A_2 = C_2(4\Lambda_K + 2)$.

It remains to show that Ω is quasisymmetrically equivalent to the disk with a controlled distortion function. To do so, we will show that $\bar{\Omega}$ is LLC and Ahlfors 2-regular with constants depending only on Λ_K and C_K. The desired result will then follow from Theorem 3.20.

We first address the LLC condition. We will not use the full strength of the fact that γ is a chord-arc loop. Instead, we will only need the weaker diameter condition given by (3.4).

Let $x \in \bar{\Omega}$ and $r > 0$. As $\bar{\Omega}$ is connected, it suffices to consider the case that $r \leq \operatorname{diam}(\bar{\Omega})$. By (4.33), the definition of A_1, and (4.31), we see that $\bar{B}_X(x, r)$ is a compact subset of U.

Suppose that $a, b \in B_{\bar\Omega}(x,r)$. Recalling that U is $2\Lambda_K$-relatively LLC, we may find a path $\alpha\colon [0,1] \to U$ such that $\alpha(0) = a$, $\alpha(1) = b$, and $\operatorname{im}\alpha \subseteq B_X(x, 2\Lambda_K r)$. Let

$$t_a = \min\{t \in [0,1] : \alpha(t) U\backslash\Omega\} \quad \text{and} \quad t_b = \max\{t \in [0,1] : \alpha(t) \in U\backslash\Omega\}.$$

Since the boundary of Ω is γ, which is a λ-chord-arc loop, there is an arc $\gamma_{ab} \subseteq \operatorname{im}\gamma$ that connects $\alpha(t_a)$ to $\alpha(t_b)$ and satisfies

$$\operatorname{diam}(\gamma_{ab}) \le \lambda d(\alpha(t_a), \alpha(t_b)) \le 4\lambda\Lambda_K r.$$

Then

$$\alpha([0,t_a]) \cup \gamma_{ab} \cup \alpha([t_b,1]) \subseteq B(x, 2\Lambda_K(4\lambda+1)r)$$

is a continuum in $\bar\Omega$ connecting a to b. This shows that $\bar\Omega$ is $2\Lambda_K(4\lambda+1)$-LLC_1.

Now suppose that $a, b \in \bar\Omega\backslash B_{\bar\Omega}(x,r)$. There is a path $\alpha\colon [0,1] \to U$ such that $\alpha(0) = a$, $\alpha(1) = b$, and $\operatorname{im}(\alpha) \subseteq U\backslash B_X(x, r/2\Lambda_K)$. Define t_a and t_b as above. We may write $\gamma = \gamma_1 \cup \gamma_2$, where γ_1 and γ_2 are closed subarcs of γ with endpoints $\alpha(t_a)$ and $\alpha(t_b)$. Suppose that we may find points

$$x_1 \in \gamma_1 \cap B_X\left(x, \frac{r}{8\Lambda_K\lambda}\right) \quad \text{and} \quad x_2 \in \gamma_2 \cap B_X\left(x, \frac{r}{8\Lambda_K\lambda}\right).$$

Then $d(x_1,x_2) \le r/(4\Lambda_K\lambda)$, and so the by the quasiarc property of γ, either $\alpha(t_a)$ or $\alpha(t_b)$ is contained in

$$\bar B\left(x_1, \frac{r}{4\Lambda_K}\right) \subseteq B\left(x, \frac{r}{2\Lambda_K}\right).$$

This contradicts the fact that $\operatorname{im}\alpha \subseteq U\backslash B_X(x, r/2\Lambda_K)$. Thus there is some $i \in \{1,2\}$ such that $\alpha([0,t_a]) \cup \gamma_i \cup \alpha([t_b,1])$ connects a to b in

$$\bar\Omega\backslash B\left(x, \frac{r}{8\Lambda_K\lambda}\right).$$

We have now shown that $\bar\Omega$ is λ'-LLC with $\lambda' = 2\Lambda_K(4\lambda+1)$.

We now show that $\bar\Omega$ is Ahlfors 2-regular with constant depending only on Λ_K and C_K. Let $x \in \bar\Omega$ and $0 \le r \le \operatorname{diam}(\bar\Omega)$. As $\bar\Omega \subseteq U$, and U is relatively Ahlfors 2-regular with constant C_K, it follows from the definition of Hausdorff measure that

$$\mathcal{H}^2_{\bar\Omega}(\bar B_{\bar\Omega}(x,r)) \le 4\mathcal{H}^2_X(\bar B_{\bar\Omega}(x,r)) \le 4\mathcal{H}^2_X(\bar B_X(x,r)) \le 4C_K r^2.$$

To show the lower bound, we consider four cases.

Case 1: $x \in \Omega$ and $r < \operatorname{dist}(x, \operatorname{im}\gamma)$. Since U is $2\Lambda_K$-relatively LLC, the x-component of $\bar B_{\bar\Omega}(x,r)$ contains $\bar B_X(x, r/4\Lambda_K)$. This implies that there is some $\epsilon > 0$ such that the ϵ-neighborhood of $\bar B_X(x, r/4\Lambda_K)$ is contained in $\bar\Omega$. As a result,

$$\mathcal{H}^2_{\bar\Omega}(\bar B_{\bar\Omega}(x,r)) \ge \mathcal{H}^2_{\bar\Omega}\left(\bar B_{\bar\Omega}\left(x, \frac{r}{4\Lambda_K}\right)\right) = \mathcal{H}^2_X\left(\bar B_X\left(x, \frac{r}{4\Lambda_K}\right)\right) \ge \frac{r^2}{C_K(4\Lambda_K)^2}.$$

Case 2: $x \in \gamma$ *and* $0 \le r \le \operatorname{diam}(\bar\Omega)$. By Theorem 4.24, we may find a number $C \ge 1$ depending only on C_K and Λ_K, and a point $y \in \Omega$ such that

$$B_{\bar\Omega}(y, r/C) \subseteq B_{\bar\Omega}(x,r)\backslash \operatorname{im}\gamma.$$

It follows from Case 1 that

$$\mathcal{H}^2_{\bar\Omega}(\bar B_{\bar\Omega}(x,r)) \ge \mathcal{H}^2_{\bar\Omega}\left(\bar B_{\bar\Omega}\left(y, \frac{r}{2C}\right)\right) \ge \frac{r^2}{C_K(8C\Lambda_K)^2}.$$

Case 3: $x \in \Omega$ and $\operatorname{dist}(x, \operatorname{im}\gamma) \le r < 4\operatorname{dist}(x, \operatorname{im}\gamma)$. By Case 1, we have

$$\mathcal{H}^2_{\bar{\Omega}}(\bar{B}_{\bar{\Omega}}(x,r)) \ge \mathcal{H}^2_{\bar{\Omega}}\left(\bar{B}_{\bar{\Omega}}\left(x, \frac{r}{4}\right)\right) \ge \frac{r^2}{C_K(16\Lambda_K)^2}.$$

Case 4: $x \in \Omega$ and $r \ge 4\operatorname{dist}(x, \operatorname{im}\gamma)$. We may find a point $y \in \gamma$ such that $d(x,y) < r/2$. Then $\bar{B}_{\bar{\Omega}}(y, r/2) \subseteq B_{\bar{\Omega}}(x,r)$, and so by Case 2 we have

$$\mathcal{H}^2_{\bar{\Omega}}(\bar{B}_{\bar{\Omega}}(x,r)) \ge \mathcal{H}^2_{\bar{\Omega}}\left(\bar{B}_{\bar{\Omega}}\left(y, \frac{r}{2}\right)\right) \ge \frac{r^2}{C_K(16C\Lambda_K)^2}.$$

Thus we have shown that $\bar{\Omega}$ is Ahlfors 2-regular with constant $64C_K(C\lambda')^2$. By (4.32) and (4.33), the ratio $\operatorname{diam}(\Omega)/\operatorname{diam}(\partial\Omega)$ depends only on C_K and Λ_K. The desired quasisymmetric homeomorphism is now provided by Theorem 3.20. □

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