Analytic properties of quasiconformal mappings between metric spaces

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Dedicated to Jeff Cheeger for his 65th birthday

Abstract. We survey recent developments in the theory of quasiconformal mappings between metric spaces. We examine the various weak definitions of quasiconformality, and give conditions under which they are all equal and imply the strong classical properties of quasiconformal mappings in Euclidean spaces. We also discuss function spaces preserved by quasiconformal mappings.

1. Introduction

A complete understanding of the behavior of quasiconformal mappings requires fluency in moving between the various definitions of quasiconformality. Of particular importance is that the Sobolev regularity and absolute continuity properties of quasiconformal mappings in fact follow from the easy to verify metric definition. In Euclidean spaces, these properties have been major theme in the literature from the initiation of the study of non-smooth quasiconformal mappings by Ahlfors in 1954 [1] to Gehring's seminal works in the early 1960's [8], [7]. By 1968, the celebrated work of Mostow demonstrated the need for a theory of quasiconformal mappings in the non-Riemannian setting [23]. This led to the study of rigidity and quasiconformal mappings in the Heisenberg group and other Carnot groups. In this setting, the techniques of Gehring, which are based on the foliation of Euclidean space by lines, become tenuous and delicate to employ, though with difficulty they still led to important results [24], [21].

An alternate approach to understanding the equivalence of the many different definitions of quasiconformal mappings was given by Heinonen and Koskela [13]. By considering quasiconformal mappings in the setting of arbitrary metric spaces, they were able to identify robust techniques that did not depend on the special structure of Euclidean spaces. The starting point is the simplest definition of a quasiconformal mapping, the *metric definition.*

For a homeomorphism $f\colon X \to Y$ of metric spaces, we define for all $x \in X$ and $r > 0$

$$L_f(x,r) := \sup\{d_Y(f(x), f(y)) : d_X(x,y) \le r\},$$

$$l_f(x,r) := \inf\{d_Y(f(x), f(y)) : d_X(x,y) \ge r\},$$

$$H_f(x) := \limsup_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)} \text{ and } h_f(x) := \liminf_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)}.$$

The mapping f is *H-quasiconformal*, $H \ge 1$, if $H_f(x) \le H$ for all $x \in X$.

If the underlying space X lacks a useful infinitesimal structure, then we cannot expect any large-scale properties of quasiconformal mappings defined on X. On the other hand, if the underlying space X has enough structure, the infinitesimal definition given above in fact guarantees strong properties of quasiconformal mappings. Properties of particular importance are Sobolev regularity, absolute continuity on paths, and quasisymmetry, i.e., global rather than infinitesimal distortion bounds. In this survey, we examine the minimal assumptions on metric spaces X and Y and a homeomorphism $f\colon X \to Y$ that guarantee that f is quasiconformal and possesses these strong properties. We also discuss recent work regarding function spaces preserved by such mappings.

2. The metric space setting

A *metric measure space* is a triple (X, d, μ) where (X, d) is a metric space and μ is a measure on X. For our purposes, a *measure* is a non-negative countably subadditive set function defined on all subsets of X that gives the value 0 to the empty set. Moreover, we require that measures are Borel inner and outer regular.

A metric space is said to be *proper* if every closed and bounded set is compact. Unless otherwise mentioned, throughout this paper we let (X, d, μ) and (Y, d_Y, ν) be proper metric measure spaces.

Given a point $x \in X$ and a radius $r > 0$, we employ the following notation for balls:

$$B_{(X,d)}(x,r) = \{y \in X : d(x,y) < r\} \quad \text{and} \quad \bar{B}_{(X,d)}(x,r) = \{y \in X : d(x,y) \le r\}.$$

Where it will not cause confusion, we will replace $B_{(X,d)}(x,r)$ by $B_X(x,r)$, $B_d(x,r)$, or $B(x,r)$. A similar convention will be made for any other objects which depend on the ambient metric space. Given a ball $B = B(x,r)$ and a constant $\tau > 0$, we denote by τB the ball $B(x, \tau r)$.

A main theme in analysis on metric spaces is that the infinitesimal structure of a metric space can be understood via the paths that it contains. The reason for this is that rectifiable paths admit path-integration. We define a *path* in X to be a continuous, non-constant map $\gamma\colon I \to X$ where $I \subseteq \mathbb{R}$ is a compact interval. A path $\gamma\colon I \to X$ is called *rectifiable* if it is of finite length. Any rectifiable path $\gamma\colon [a,b] \to X$ has a unique parameterization $\gamma_s\colon [0, \text{length}(\gamma)] \to X$ such that for all $t \in [a,b]$,

$$\gamma(t) = \gamma_s(\text{length}(\gamma|_{[a,t]})).$$

The path γ_s is called the *arc length parameterization of* γ, and it is 1-Lipschitz. Given a Borel function $\rho\colon X \to [0,\infty]$ and a rectifiable path γ in X, we define the integral of ρ over γ by

$$\int_\gamma \rho\ ds := \int_{[0,\text{length}(\gamma)]} \rho \circ \gamma_s(t)\ dt.$$

A measurement of the size of a given collection of paths Γ in X is the *p-modulus of* Γ, $p \geq 1$, which is defined by

$$\text{mod}_p(\Gamma) = \inf \int_X \rho^p\ d\mu,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0,\infty]$ such that for all locally rectifiable paths $\gamma \in \Gamma$,

$$\int_\gamma \rho\ ds \geq 1.$$

Such a function ρ is said to be *admissible* for the path family Γ. A condition is said to be true on *p-almost every path in X* if the collection of paths in X where the condition does not hold has p-modulus 0.

An upper gradient of f is a generalization of the norm of the gradient of f developed in connection with quasiconformal mappings in [13]. Philosophically, the more rectifiable curves a metric space contains, the more stringent the upper gradient condition becomes. Given an open set $U \subseteq X$ and a mapping $f\colon U \to Y$, we say that a Borel function $\rho\colon U \to [0,\infty]$ is an *upper gradient of f* in U if, for each rectifiable path $\gamma\colon [0,1] \to U$, we have

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma \rho\ ds. \tag{2.1}$$

If (2.1) holds only for p-almost every path in U, then we say that ρ is a *p-weak upper gradient* of f in U.

Real-valued Sobolev spaces based on upper gradients were used to great success [6] and explored in-depth in [27]. They have been extended to the metric-valued setting in [14] and have seen numerous generalizations. A simple definition is as follows. Let $f\colon X \to Y$ be a continuous map. Then f is in the Sobolev space $W^{1,p}_{\text{loc}}(X;Y)$, $1 \leq p \leq \infty$, if for each relatively compact open subset $U \subseteq X$, the map f has an upper gradient $g \in L^p(U)$ in U, and there is a point $x_0 \in U$ such that $u(x) := d_Y(f(x_0), f(x)) \in L^p(U)$. If the space Y is not specified, it assumed to be $\mathbb{R}$.

A continuous mapping $f\colon X \to Y$ is said to be *absolutely continuous on a rectifiable path γ in X* if the map $f \circ \gamma_s\colon [0, \text{length}(\gamma)] \to Y$ is absolutely continuous in the usual sense. As in the Euclidean setting, Sobolev maps of metric spaces (which are defined to be continuous) have absolute continuity properties. Namely, if X is proper, then each $f \in W^{1,p}_{loc}(X;Y)$ is absolutely continuous on p-almost every rectifiable path in X [27, Prop. 3.1].

In the Euclidean setting, the dimension of the space obviously plays a key role in the Sobolev theory. Moreover, the dimension of the space is reflected in the uniform scaling of Lebesgue measure. The following condition is a relatively strong generalization of this phenomenon to the metric measure space setting. The metric measure space (X,d,μ) is called *Ahlfors Q-regular*, $Q \geq 0$, if there exists a constant $K \geq 1$ such that for all $a \in X$ and $0 < r \leq \operatorname{diam} X$, we have

$$\frac{r^Q}{K} \leq \mu(\bar{B}_d(a,r)) \leq K r^Q. \tag{2.2}$$

We say that (X,d,μ) is *locally Ahlfors Q-regular* if for every compact subset $V \subseteq X$, there is a constant $K \geq 1$ and a radius $r_0 > 0$ such that for each point $a \in V$ and radius $0 < r \leq r_0$, the inequalities in (2.2) are satisfied.

We also require the following non-standard definition. Let $E \subseteq X$. We say that (X,d,μ) is *locally Ahlfors Q-regular off E* if there is a constant $K \geq 1$ such that for each point $a \in X \backslash E$, there is a radius $r_a > 0$ such that for each $0 < r \leq r_a$, the inequalities in (2.2) are satisfied.

There is also a weaker notion of dimensionality for measures that is useful. The metric measure space (X,d,μ) is *doubling* if there is a constant $C \geq 1$ such that for every $x \in X$ and $r > 0$,

$$\mu(B(x,2r)) \leq C\mu(B(x,r)).$$

Iterating this condition leads to the notion of *Assouad dimension*; for more information, see for example [12].

The space (X,d,μ) is said to support a *p-Poincaré inequality*, $1 \leq p < \infty$ if there are constants $C, \tau \geq 1$ such that if B is a ball in X, $u\colon \tau B \to \mathbb{R}$ is a bounded continuous function, and ρ is an upper gradient of u, then

$$\fint_B |u - u_B| \, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\tau B} \rho^p \, d\mu \right)^{1/p}.$$

Here and throughout the paper we employ the notation

$$u_B = \fint_B u \, d\mu = \frac{1}{\mu(B)} \int_B u \, d\mu,$$

whenever u is a μ-measurable function on B.

Note that if (X,d,μ) supports a p-Poincaré inequality, $1 \leq p < \infty$, then it also supports a q-Poincaré inequality for all $q \geq p$. A deep theorem of Keith and Zhong states that if (X,d,μ) is doubling and supports a p-Poincaré inequality, $p > 1$, then it also supports a p'-Poincaré inequality for some $p' < p$ [16].

The p-Poincaré inequality can be thought of as a requirement that a space contains "many" curves, in terms of the p-modulus of curves in the space. See [13], [12], and [15] for more information. For a small bit of intuition, assume that (X,d,μ) is Ahlfors Q-regular, $Q > 1$, and supports a p-Poincaré inequality. Regardless of the value of p, it follows that (X,d) is quasiconvex.

However, if $p > Q$, then X may contain local cut-points. For more geometric implications of the Poincaré inequality, see [11] and [17].

3. Weak definitions of quasiconformality

In this section, we discuss weak versions of the metric, analytic, and geometric definitions of quasiconformality. Some relations between these conditions are valid even in the absence of a Poincaré inequality. Their value resides in the fact that in the presence of an appropriate Poincaré inequality, they are all equivalent to the usual strong forms of quasiconformality. However, they are potentially much easier to verify in practice.

We begin with a weak formulation of the metric definition. The following definition allows for an exceptional set and employs h_f rather than H_f.

Definition 3.1. Let $1 \le p \le Q$. We say that f satisfies the *weak* (Q,p)*-metric definition of quasiconformality* if one of the following two conditions holds:

- $p < Q$, and there exists a set $E \subseteq X$ of σ-finite $(Q-p)$-dimensional Hausdorff measure, and a number $0 \le H < \infty$ such that $h_f(x) < \infty$ for all $x \in X \backslash E$ and $h_f(x) \le H$ for μ-almost every point $x \in X$,
- $p = Q$, and there exists a countable set $E \subseteq X$ and a number $0 \le H < \infty$ such that $h_f(x) \le H$ for all $x \in X \backslash E$

The classical definition of a quasiconformal homeomorphism $f\colon \Omega \to \Omega'$ of domains in $\mathbb{R}^n$ consists of the requirements that $f \in W^{1,n}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$ and that there is a constant $K \ge 1$ such that $||Df||^n \le K J_f$ almost everywhere, where Df is the weak differential matrix of f and J_f is the determinant of Df. Given a homeomorphism $f\colon (X,d,\mu) \to (Y,d_Y,\nu)$, the role of J_f is played by the *volume derivative* $\mu_f\colon X \to [0,\infty]$ defined by

$$\mu_f(x) = \limsup_{r\to 0} \frac{\nu(f(B(x,r)))}{\mu(B(x,r))}.$$

Definition 3.2. We say that f satisfies the Q*-analytic definition of quasiconformality* if $f \in W^{1,Q}_{\mathrm{loc}}(X;Y)$ and there is a constant $H \ge 1$ such that $H\mu_f^{1/Q}$ is a Q-weak upper gradient of f.

Finally, we consider the geometric (or modulus) definition of quasiconformality, first introduced by Ahlfors in Euclidean space [1]. Many properties of quasiconformal mappings may be derived directly from this definition.

Definition 3.3. We say that f satisfies the *weak* Q*-geometric definition of quasiconformality* if there is a constant $H \ge 1$ such that for every path family Γ in X

$$\operatorname{mod}_Q(\Gamma) \le H \operatorname{mod}_Q(f(\Gamma)).$$

If f and f^{-1} satisfy the weak Q-geometric definition of quasiconformality, then we say that f satisfies the Q*-geometric definition of quasiconformality.*

The key point in connecting these definitions is absolute continuity on paths. The following result from [2] establishes absolute continuity on p-almost every path for mappings satisfying the weak (Q,p)-metric definition of quasiconformality.

Theorem 3.4 (Balogh-Koskela-Rogovin). *Let $Q > 1$ and let $1 \leq p \leq Q$. Suppose f satisfies the weak (Q,p)-metric definition of quasiconformality, and assume that X is locally Ahlfors Q-regular and that Y is locally Ahlfors Q-regular off $f(E)$. Then $f \in W^{1,p}_{\mathrm{loc}}(X;Y)$.*

The relationship between the weak metric and analytic definitions of quasiconformality is now provided by the following results.

Theorem 3.5 (Balogh-Koskela-Rogovin). *Assume the hypotheses of Theorem* 3.4*. Then*

$$g_f(x) := \begin{cases} H(\mu_f(x))^{1/Q} & h_f(x) \leq H, \\ \infty & h_f(x) > H, \end{cases}$$

is a p-weak upper gradient of f.

Corollary 3.6. *Assume the hypotheses of Theorem* 3.4 *with $p = Q$. Then f satisfies the Q-analytic definition of quasiconformality.*

The Q-analytic definition and the weak Q-geometric definition are closely linked. Note that in the following result from [31], no form of Ahlfors Q-regularity is assumed.

Theorem 3.7 (Williams). *Assume that (X, d_X, μ) and (Y, d_Y, ν) are separable metric measure spaces of locally finite measure, and that (X, d, μ) is doubling. Then f satisfies the Q-analytic definition of quasiconformality if and only if it satisfies the weak Q-geometric definition.*

Many properties of quasiconformal mappings between Ahlfors Q-regular metric spaces follow directly from the (strong) Q-geometric definition. Thus it is of practical interest to understand the properties of the inverse of a mapping that satisfies a weak definition of quasiconformality.

Theorem 3.8. *Assume that (X, d_X, μ) and (Y, d_Y, ν) are locally Ahlfors Q-regular, $Q > 1$. If $f\colon X \to Y$ satisfies the weak (Q,Q)-metric definition of quasiconformality, then $f^{-1} \in W^{1,Q}_{\mathrm{loc}}(X;Y)$. Moreover, f^{-1} satisfies the Q-analytic definition of quasiconformality, and f satisfies the Q-geometric definition of quasiconformality.*

Proof. The proof is nearly identical to the proof of Theorem 3.4 given in [2, Theorem 4.1]; the main philosophical difference is that one replaces balls with ball-like objects, namely, the images of balls under f. One constructs the same cover as in that proof, but replace the control function ρ_ϵ defined there by the the quantity

$$\rho_\epsilon(y) = \sum_i \frac{r_i}{L_f(x_i, r_i)} \chi_{B(f(x_i), 2L_f(x_i, r_i))}(y).$$

The remainder of the proof procedes as in the original to show that $f^{-1} \in W^{1,Q}_{\mathrm{loc}}(X;Y)$. A similar trick applied to the proof of Theorem 3.5 given in [2, Proposition 4.3] shows that f^{-1} satisfies the Q-analytic definition of quasiconformality. The final statement now follows from Corollary 3.6 and Theorem 3.7. □

4. Quasiconformality and quasisymmetry

In the Euclidean setting, the natural Sobolev regularity of a quasiconformal mapping corresponds to the dimension of the space. It turns out that the assumption of a suitable Poincaré inequality allows for similar results in the metric space setting, as the following modified version of Theorem 3.4 shows.

Theorem 4.1. *Let $Q > 1$ and $1 \leq p \leq Q$. Suppose f satisfies the weak (Q,p)-metric definition of quasiconformality, and assume that X is locally Ahlfors Q-regular and satisfies a p-Poincaré inequality, and that Y is locally Ahlfors Q-regular off $f(E)$. Then $f \in W^{1,Q}_{\mathrm{loc}}(X;Y)$.*

It has been shown that Theorem 3.4 is also sharp in the sense that in the absence of a Poincaré inequality, the absolute continuity and Sobolev regularity cannot be improved to the Euclidean analogue [18].

Theorem 4.2. *For each integer $m \geq 1$ and real number $\epsilon > 0$, there is a homeomorphism $f\colon X \to Y$ of metric measure spaces and a set $E \subseteq X$ such that*

- (i) *X is compact, quasiconvex, and Ahlfors 2-regular,*
- (ii) *Y is compact and locally Ahlfors 2-regular off $f(E)$,*
- (iii) *$(\log_3 2)/m \leq \dim_H(E) \leq (2\log_3 2)/m$, and $0 < \mathcal{H}^{\dim_H(E)}(E) < \infty$,*
- (iv) *$H_f(x) = 1$ for all $x \in X \backslash E$,*
- (v) *$f \notin W^{1,q}_{loc}(X;Y)$ for some $q < 2 - \dim_H(E) + \epsilon$.*

The abundance of curves provided by a Poincaré inequality also allows for much stronger *global* distortion estimates in the form of quasisymmetry, at least in the presence of necessary geometric conditions.

A homeomorphism $f\colon (X, d_X) \to (Y, d_Y)$ of metric spaces is called *quasisymmetric* if there exists a homeomorphism $\eta\colon [0,\infty) \to [0,\infty)$ such that for all triples $a, b, c \in X$ of distinct points, we have

$$\frac{d_Y(f(a), f(b))}{d_Y(f(a), f(c))} \leq \eta\left(\frac{d_X(a,b)}{d_X(a,c)}\right).$$

It is easy to see that such a homeomorphism satisfies $L_f(x,r) \leq \eta(1) l_f(x,r)$ for all $x \in X$ and $r \geq 0$. If f is a quasisymmetric homeomorphism, then f^{-1} is as well; indeed, most proofs that the inverse of a quasiconformal mapping is quasiconformal involve first showing that the map is quasisymmetric.

As opposed to quasiconformal mappings, quasisymmetric mappings preserve boundedness and prevent the formation of cusps in a space. The latter property can be formalized as follows.

Let $\lambda > 1$. A metric space (X, d) is *λ-linearly locally connected* (λ-*LLC*) if for all $a \in X$ and $r > 0$ the following conditions are satisfied:

(λ-LLC_1) For each pair of distinct points $x, y \in B(a, r)$, there is a continuum $E \subseteq B(a, \lambda r)$ such that $x, y \in E$,

(λ-LLC_2) For each pair of distinct points $x, y \in X \backslash B(a, r)$, there is a continuum $E \subseteq X \backslash B(a, r/\lambda)$ such that $x, y \in E$.

Recall that a continuum is a connected, compact set containing more than one point. If $f\colon X \to Y$ is an η-quasisymmetric homeomorphism, and X is λ-LLC, then Y is λ'-LLC where λ' depends only on η and λ. If a metric measure space (X, d, μ) is Ahlfors Q-regular and supports a Q-Poincaré inequality, then it is λ-LLC for some $\lambda \geq 1$ depending only on the data associated to the conditions on the space [13].

The following theorem may now be derived from [2, Theorem 5.1 and Remark 5.3] and the techniques of the proof of Theorem 3.8.

Theorem 4.3 (Balogh-Koskela-Rogovin). *Suppose that X and Y are Ahlfors Q-regular metric spaces that are simultaneously bounded or unbounded, and that one of X and Y is linearly locally connected and the other satisfies a Q-Poincaré inequality. If $f\colon X \to Y$ satisfies the weak (Q, Q)-metric definition of quasiconformality, then $f \in W^{1,Q}_{\mathrm{loc}}(X; Y)$ and f is quasisymmetric. Moreover, f satisfies the Q-analytic and Q-geometric definitions of quasiconformality, and is absolutely continuous in measure.*

It is not true that quasisymmetric mappings must be absolutely continuous in measure, in the absence of a Poincaré inequality.

Example 4.4. For any $Q \geq 1$, there is an Ahlfors Q-regular and LLC metric measure space (X, d_X, μ) and a quasisymmetric mapping $f\colon X \to X$ such that f maps a set of measure zero to a set of full measure and a set of full measure to a set of zero measure.

Proof. It was shown by Tukia [29] that there exists an η-quasisymmetric mapping $g\colon \mathbb{R} \to \mathbb{R}$ mapping a set of measure zero to a set of full measure and vice-versa. Let $X = \mathbb{R}$, $d_X = |\cdot|^{1/Q}$, and $\mu = \mathcal{H}^Q_{d_X}$. Then (X, d_X, μ) is Q-regular, and for any $E \subseteq \mathbb{R}$, the quantity $\mu(E)$ is equal to the one-dimensional Lebesgue measure of E. Moreover, g is also a quasisymmetric when considered as a mapping from (X, d_X) to itself, with distortion function $\widetilde{\eta}(t) = (\eta(t^Q))^{1/Q}$. $\square$

The most general setting in which even very strong definitions of quasiconformality imply quasisymmetry is not clear. The question is particularly intriguing in infinite dimensions, even for very simple function spaces.

Theorem 4.5 (Naor). *The space L_p quasisymmetrically embeds in the space L_2 if and only if $p \leq 2$.*

Question 4.6 (Naor). Does L_p quasiconformally embed in L_2 when $p > 2$?

The proof of Theorem 4.5 relies on deep work of Mendel and Naor that gives a metric characterization of Rademacher cotype in Banach spaces [22]. In particular, it is shown that cotype is preserved by quasisymmetric mappings between K-convex Banach spaces [25]. The study of cotype in the general non-linear setting is still nascent [30].

While quasiconformal mappings on $\mathbb{R}^n$ are defined to be in $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n)$, each such mapping actually lies in a smaller Sobolev class $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$, where $p > n$ depends n and the distortion of the individual mapping [5], [9]. The key fact behind this improved regularity is the following *reverse Hölder inequality*, which has a self-improving property. For $t \in [1, \infty]$, we say that a real-valued function $u\colon X \to \mathbb{R}$ is in the class $\mathcal{B}_t(X)$ if there is a quantity $C \geq 1$ such that for every $x \in X$ and $r > 0$,

$$\left(\fint_{B(x,r)} \mu_f^t \, d\mu \right)^{1/t} \leq C \fint_{B(x,r)} \mu_f \, d_\mu$$

when $t < \infty$, or an analogous condition when $t = \infty$. For each quasiconformal mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$, the Jacobian determinant J_f "naturally" lies in the class $\mathcal{B}_1(\mathbb{R}^n)$, but in fact lies in a smaller class $\mathcal{B}_t(\mathbb{R}^n)$ for some $t > 1$ that depends on n and the distortion of the individual mapping. The following result shows that this phenomenon persists in the presence of an appropriate Poincaré inequality (Theorem 4.3, [13], [16]).

Theorem 4.7. *Assume the hypotheses of Theorem* 4.3*. Then there is $t > 1$ such that $\mu_f \in \mathcal{B}_t(X)$.*

5. Function spaces preserved by quasisymmetric mappings

A fundamental problem in the theory of quasiconformal mappings between metric spaces is determining when a metric space that is topologically equivalent to a "model" space (such as $\mathbb{S}^n$) is actually quasisymmetrically equivalent to that model space. This problem is of particular importance in geometric group theory [3]. One approach to this problem is to find a function space associated to each metric space that is preserved under quasisymetric mappings.

The uniformization problem described above is of interest even when the given space is not known to possess any rectifiable curves. An approach to gradients on metric spaces that does not rely on path integration was explored in [11] and [10]. Given a mapping $f\colon (X, d_X, \mu) \to (Y, d_Y)$, a measurable function $g\colon X \to [0, \infty]$ is a *Hajłasz gradient* of f if for almost every $x, y \in X$,

$$d_Y(f(x), f(y)) \leq d_X(x, y)(g(x) + g(y)).$$

This definition is both local and global in nature. One should view a Hajłasz gradient, in the Euclidean setting or in the presence of a suitable Poincaré inequality, as the maximal function of the usual gradient.

More precise variants of the concept of a Hajłasz gradient have led to interesting function spaces that are invariant under quasisymmetric mappings [20].

Definition 5.1. Let $s \in (0, \infty)$ and u be a measurable function on X. A sequence of nonnegative measurable functions, $\vec{g} \equiv \{g_k\}_{k\in\mathbb{Z}}$, is called a *fractional s-Hajłasz gradient* of u if there exists $E \subset X$ with $\mu(E) = 0$ such that for all $k \in \mathbb{Z}$ and $x, y \in X \setminus E$ satisfying $2^{-k-1} \le d(x, y) < 2^{-k}$,

$$|u(x) - u(y)| \le [d(x, y)]^s [g_k(x) + g_k(y)]. \tag{5.1}$$

Denote by $\mathbb{D}^s(u)$ the *collection of all fractional s-Hajłasz gradients of u.*

Relying on this concept we now introduce counterparts of Triebel-Lizorkin spaces. Let $p \in (0, \infty)$. In what follows, for $q \in (0, \infty]$, we always write $\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^q} \equiv \{\sum_{j\in\mathbb{Z}} |g_j|^q\}^{1/q}$ when $q < \infty$ and $\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^\infty} \equiv \sup_{j\in\mathbb{Z}} |g_j|$,

$$\|\{g_j\}_{j\in\mathbb{Z}}\|_{L^p(X,\,\ell^q)} \equiv \|\|\{g_j\}_{j\in\mathbb{Z}}\|_{\ell^q}\|_{L^p(X)}.$$

Definition 5.2. Let $s, p \in (0, \infty)$ and $q \in (0, \infty]$. The *homogeneous Hajłasz-Triebel-Lizorkin space* $\dot{M}^s_{p,\,q}(X)$ is the space of all measurable functions u such that

$$\|u\|_{\dot{M}^s_{p,\,q}(X)} \equiv \inf_{\vec{g}\in\mathbb{D}^s(u)} \|\vec{g}\|_{L^p(X,\,\ell^q)} < \infty.$$

Theorem 5.3 (Bourdon-Pajot). *Let X_1 and X_2 be Ahlfors Q_1-regular and Q_2-regular spaces with $Q_1, Q_2 \in (0, \infty)$, respectively. Let f be a quasisymmetric mapping from X_1 onto X_2. For $s_i \in (0, Q_i)$ with $i = 1, 2$, if $Q_1/s_1 = Q_2/s_2$, then f induces an equivalence between $\dot{M}^{s_1}_{Q_1/s_1,\,Q_1/s_1}(X_1)$ and $\dot{M}^{s_2}_{Q_2/s_2,\,Q_2/s_2}(X_2)$.*

In [4], Bourdon and Pajot proved the above invariance for the Besov spaces $\dot{\mathcal{B}}^{s_i}_{Q_i/s_i}$, consisting of all measurable u with

$$\|u\|_{\dot{\mathcal{B}}^{s_i}_{Q_i/s_i}(X_i)} \equiv \left(\int_{X_i}\int_{X_i} \frac{|u(x) - u(y)|^{Q_i/s_i}}{[d(x, y)]^{2Q}}\, d\mu(y)\, d\mu(x)\right)^{s_i/Q_i} < \infty.$$

The fact that these spaces coincide with the above spaces $\dot{M}^s_{p,\,q}(X)$ for the indicated indices was established in [20] together with more general invariance properties described below.

It is not claimed in Theorem 5.3 that f acts as a composition operator, but merely that every $u \in \dot{\mathcal{B}}^{s_2}_{Q_2/s_2}(X_2)$ has a representative $\widetilde{u}$ so that $\widetilde{u} \circ f \in \dot{\mathcal{B}}^{s_1}_{Q_1/s_1}(X_1)$ with a norm bound, and similarly for f^{-1}. Indeed, $u \circ f$ need not even be measurable in this generality.

If both f and f^{-1} are absolutely continuous in measure and $\mu_f \in \mathcal{B}_s(X)$ for some $s \in (1, \infty]$, then the third index in Theorem 5.3 may be replaced by an arbitrary $q > 0$. In particular, these conditions are met under the assumptions of Theorem 4.3.

Theorem 5.4. *Let X_1 and X_2 be Ahlfors Q_1-regular and Q_2-regular spaces with $Q_1, Q_2 \in (0, \infty)$, respectively. Let f be a quasisymmetric mapping from X_1 onto X_2, and assume that f and f^{-1} are absolutely continuous and $\mu_f \in$*

$\mathcal{B}_t(X_1)$ for some $t \in (1, \infty]$. Let $s_i \in (0, Q_i)$ with $i = 1, 2$ satisfy $Q_1/s_1 = Q_2/s_2$, and $q \in (0, \infty]$. Then f induces an equivalence between $\dot{M}^{s_1}_{Q_1/s_1,\, q}(X_1)$ and $\dot{M}^{s_2}_{Q_2/s_2,\, q}(X_2)$.

In Theorem 5.4, f and f^{-1} act as composition operators. Moreover, with the assumptions of Theorem 5.4, by Lebesgue-Radon-Nykodym Theorem and [28], we have that $\mu_{f^{-1}}(y) = [\mu_f(f^{-1}(y))]^{-1}$ for almost all $y \in X_2$, and hence $\mu_{f^{-1}} \in \mathcal{B}_{t'}(X_2)$ for some $t' \in (1, \infty]$

It is immediate from the definition that $u \in \dot{M}^s_{Q/s,\,\infty}(X)$ if and only if there is a set $E \subset X$ with $\mu(E) = 0$ and $g \in L^{Q/s}(X)$ such that for all $x, y \in X \setminus E$,

$$|u(x) - u(y)| \le [d(x, y)]^s [g(x) + g(y)]. \tag{5.2}$$

That is, u belongs to the space $\dot{M}^s_{Q/s}(X)$ introduced by Hajłasz in [10]. Thus these spaces are invariant in the setting of Theorem 5.4. If we further assume that X satisfies a Q-Poincaré inequality, then $\dot{M}^1_Q(X) = \dot{W}^{1,Q}(X)$ (see [16]), and thus Theorem 5.4 includes the invariance of this space under the Poincaré inequality assumption, generalizing [19]. In fact, the invariance of this space holds already under the assumptions of Theorem 3.8.

We further point out that the class of functions of bounded mean oscillation is invariant under quasisymmetric mappings of $\mathbb{R}^n$, $n \ge 2$ [26]. This space is also invariant under the assumptions of Theorem 5.4.

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