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Exceptional Sets for Quasiconformal Mappings in General Metric Spaces

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A theorem of Balogh, Koskela, and Rogovin states that in Ahlfors Q-regular metric spaces which support a p-Poincaré inequality, $1 \leq p \leq Q$, an exceptional set of σ-finite $(Q-p)$-dimensional Hausdorff measure can be taken in the definition of a quasiconformal mapping while retaining Sobolev regularity analogous to that of the Euclidean setting. Through examples, we show that the assumption of a Poincaré inequality cannot be removed.

In memoriam: Juha Heinonen (1960–2007)

1 Introduction

Classically, a homeomorphism $f\colon \Omega \to \Omega'$ of domains in $\mathbb{R}^n$, $n \geq 2$, is said to be quasiconformal if $f \in W^{1,n}_{loc}(\Omega, \Omega')$, and there is a constant $K \geq 1$ such that $\|Df(x)\|^n \leq K J_f(x)$ for almost every $x \in \Omega$. In particular, f is absolutely continuous on n-almost every path in Ω.

For a homeomorphism $f\colon X \to Y$ of metric spaces, we define for all $x \in X$ and $r > 0$

$$L_f(x,r) := \sup\{d_Y(f(x), f(y)) : d_X(x,y) \leq r\},$$
$$l_f(x,r) := \inf\{d_Y(f(x), f(y)) : d_X(x,y) \geq r\},$$
$$H_f(x) := \limsup_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)} \quad \text{and} \quad h_f(x,r) := \liminf_{r\to 0} \frac{L_f(x,r)}{l_f(x,r)}.$$

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In the 1960's, Gehring [3, 4] and Väisälä [16] gave the following metric characterization of quasiconformality in Euclidean space.

Theorem 1.1 ([3]). An orientation preserving homeomorphism $f\colon \Omega \to \Omega'$ of domains in $\mathbb{R}^n$, $n \geq 2$, is quasiconformal if and only if there is a constant $H \geq 1$ and a set $E \subseteq \Omega$ of σ-finite $(n-1)$-dimensional Hausdorff measure such that $H_f(x) < \infty$ for all $x \in \Omega \backslash E$ and $H_f(x) \leq H$ for almost every point $x \in \Omega$. □

Theorem 1.1 led to the definition of a quasiconformal mapping on Carnot groups and more general metric spaces. According to [7], a homeomorphism $f\colon X \to Y$ of metric spaces is quasiconformal if there is a constant $H \geq 1$ such that $H_f(x) \leq H$ for all $x \in X$. One fact that makes the study of such mappings a viable and rich field is that in sufficiently nice metric spaces, they possess regularity similar to that of quasiconformal mappings in Euclidean space. The following theorem, which is now well known, can be deduced from [7, 9].

Theorem 1.2 ([6, 9]). A quasiconformal homeomorphism between bounded, Ahlfors Q-regular metric spaces which support a Q-Poincaré inequality, $Q > 1$, is in $W^{1,Q}_{loc}$ and hence absolutely continuous on Q-almost every curve. □

For related results in the group setting, see [12–14], and [10]. The severity of the assumptions on the mapping in Theorem 1.2 can be reduced, assuming the ambient space is Euclidean [6, 8]. Balogh, Koskela, and Rogovin generalized these results to a metric setting where no Poincaré inequality is assumed [1].

Theorem 1.3 ([1]). Let (X, d_X, μ) be a proper, locally Ahlfors Q-regular metric measure space, $Q > 1$, and suppose that $E \subseteq X$ has σ-finite $(Q-p)$-dimensional Hausdorff measure for some $1 \leq p < Q$. Let $f\colon X \to Y$ be a homeomorphism to a metric measure space (Y, d_Y, ν) such that Y is proper and locally Ahlfors Q-regular off $f(E)$. If there is a constant $H < \infty$ with $h_f(x) < \infty$ for all $x \in X \backslash E$ and $h_f(x) \leq H$ for μ-almost every point $x \in X$, then $f \in W^{1,p}_{loc}(X; Y)$. In particular, f is absolutely continuous on p-almost every rectifiable path in X. □

Theorem 1.3 generalizes Theorem 1.2 in three major ways. First, it demands a bound only on h_f, rather than on H_f. Second, it allows for an exceptional set. Third, it does not require that the metric spaces support a Poincaré inequality. The price to pay for this is a reduction in the regularity achieved. Specifically, the regularity provided by

Theorem 1.3 is $W^{1,p}_{loc}(X;Y)$ rather than $W^{1,Q}_{loc}(X;Y)$. This is due to combination of the second and third generalizations listed above, as Theorem 1.2 and the following theorem from [1] show.

Theorem 1.4 ([1]). Assume the hypotheses of Theorem 1.3, and further assume that X supports a p-Poincaré inequality. Then $f \in W^{1,Q}_{loc}(X;Y)$. □

In the statements of Theorems 1.3 and 1.4 in [1], the target space Y was assumed to be locally Ahlfors Q-regular rather than locally Ahlfors Q-regular off $f(E)$, but the proofs given therein provide the slightly more general versions stated above. We defer the precise definitions to Section 2.

Our main result shows that the loss of regularity in Theorem 1.3 is unavoidable in the general metric setting, and is not an artifact of the proof of Theorem 1.3.

Theorem 1.5. For all $\alpha > 0$ and $\epsilon > 0$, there is a homeomorphism $f\colon X \to Y$ of metric measure spaces and a set $E \subseteq X$ such that

(i) X is compact, quasiconvex, and Ahlfors 2-regular,
(ii) Y is compact and locally Ahlfors 2-regular off $f(E)$,
(iii) $\dim_H(E) \leq \alpha$, and $0 < \mathcal{H}^{\dim_H(E)}(E) < \infty$,
(iv) $H_f(x) = 1$ for all $x \in X \backslash E$,
(v) $f \notin W^{1,q}_{loc}(X;Y)$ for some $q < 2 - \dim_H(E) + \epsilon$. □

The conclusions (i)–(iv) above fulfill the hypotheses of Theorem 1.3, so we see that f as above is in the space $W^{1,2-\dim_H(E)}_{loc}(X;Y)$. By Theorem 1.4, if X supported a $2-\dim_H(E)$-Poincaré inequality, then f would have to be in the space $W^{1,2}_{loc}(X;Y)$. Our result shows that this cannot be the case. We suspect that X supports a p-Poincaré inequality for some $p < 2 - \dim_H(E) + \epsilon$, but due to the technical nature of the construction, we are unable to provide a concise proof of this. It would also be interesting to have an example fulfilling the requirements of Theorem 1.5 where Y is globally Ahlfors 2-regular. In this paper, we focus on dimension 2 only for simplicity—similar constructions can be made in any integral dimension greater than 1.

Our construction is quite concrete and is in the spirit of the following classical example, which shows that the size of the exceptional set in Theorem 1.1 cannot be increased. Let $\Omega = (0,1) \times \mathbb{R}$, let $\mathcal{C}$ be a regular Cantor set in $[0,1]$ of dimension $0 < \epsilon < 1$, and let $c\colon [0,1] \to [0,1]$ be the corresponding Cantor function. Define $f\colon \Omega \to \Omega$ by $f(x,y) = (x, y + c(x))$. Then $H_f = 1$ except on $\mathcal{C} \times \mathbb{R}$, which is a set of σ-finite

$(1+\epsilon)$-dimensional Hausdorff measure. However, f is not absolutely continuous on any horizontal line traversing Ω. The family of such lines has positive 2-modulus, and so f is not in $W^{1,2}_{loc}(\Omega, \Omega)$ and therefore does not satisfy the analytic definition of quasiconformality.

2 Notation, Definitions, and Basic Facts

Throughout this section, let (X, d_X, μ) and (Y, d_Y, ν) be metric measure spaces. The concepts we will introduce are fairly standard. A more complete discussion can be found in [5, 7].

Given a point $x \in X$ and a radius $r > 0$, we employ the following notation for balls:

$$B_{(X,d)}(x,r) = \{y \in X : d(x,y) < r\} \quad \text{and} \quad \overline{B}_{(X,d)}(x,r) = \{y \in X : d(x,y) \leq r\}.$$

A metric space is said to be proper if every closed and bounded set is compact.

Where it will not cause confusion, we will replace $B_{(X,d)}(x,r)$ by $B_X(x,r)$, $B_d(x,r)$, or $B(x,r)$. A similar convention will be made for any other objects, which depend on the ambient metric space. If $\tau > 0$, and $B = B(x,r)$ is a ball, then we set $\tau B = B(x, \tau r)$. For $\epsilon > 0$ and $E \subseteq X$, we denote

$$\mathcal{N}_\epsilon(E) = \bigcup_{x \in E} B(x,\epsilon) \quad \text{and} \quad \overline{\mathcal{N}}_\epsilon(E) = \bigcup_{x \in E} \overline{B}(x,\epsilon).$$

A homeomorphism $f\colon X \to Y$ is called an s-similarity, $s > 0$, if for all $x, y \in X$,

$$d_Y(f(x), f(y)) = s d_X(x,y).$$

If there is a constant $L \geq 1$ such that for all $x, y \in X$,

$$\frac{d_X(x,y)}{L} \leq d_Y(f(x), f(y)) \leq L d_X(x,y), \tag{2.1}$$

then f is called bi-Lipschitz. If only the second inequality in (2.1) is assumed to hold, then f is called Lipschitz.

We denote the length of an interval $I \subseteq \mathbb{R}$ by $|I|$. We define a path in X to be a continuous, nonconstant map $\gamma\colon I \to X$ where $I \subseteq \mathbb{R}$ is a compact interval. A path $\gamma : I \to X$ is called rectifiable if it is of finite length. The space X is called quasiconvex if there is a constant $L \geq 1$ such that for each pair of points $x, y \in X$, there is a path in X connecting x to y of length no greater than $Ld(x, y)$.

Any rectifiable path $\gamma\colon [a, b] \to X$ has a unique parameterization $\gamma_s\colon [0, \mathrm{length}(\gamma)] \to X$ such that for all $t \in [a, b]$,

$$\gamma(t) = \gamma_s(\mathrm{length}(\gamma|_{[a,t]})).$$

The path γ_s is called the arc length parameterization of γ, and it is 1-Lipschitz.

Given a Borel function $\rho\colon X \to [0, \infty]$ and a rectifiable path γ in X, we define the integral of ρ over γ by

$$\int_\gamma \rho \, ds := \int_{[0,\mathrm{length}(\gamma)]} \rho \circ \gamma_s(t) \, dt.$$

Given a collection of paths $\Gamma \subseteq X$, we define the p-modulus of Γ, $p > 1$, by

$$\mathrm{mod}_p(\Gamma) = \inf \int_X \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho\colon X \to [0, \infty]$ such that for all locally rectifiable paths $\gamma \in \Gamma$,

$$\int_\gamma \rho \, ds \geq 1.$$

Such a function ρ is said to be admissible for the path family Γ. A condition is said to be true on p-almost every path in X if the collection of paths in X where the condition does not hold has p-modulus 0.

Given an open set $U \subseteq X$ and a mapping $f\colon U \to Y$, we say that a Borel function $\rho\colon U \to [0, \infty]$ is an upper gradient of f in U if, for each rectifiable path $\gamma\colon [0, 1] \to U$, we have

$$d_Y(f(\gamma(0)), f(\gamma(1))) \leq \int_\gamma \rho \, ds.$$

Let $f\colon X \to Y$ be a continuous map. Then f is in the Sobolev space $W^{1,p}_{loc}(X; Y)$, $1 \leq p \leq \infty$, if for each relatively compact open subset $U \subseteq X$, the map f has an upper gradient $g \in L^p(U)$ in U, and there is a point $x_0 \in U$ such that $u(x) := d_Y(f(x_0), f(x)) \in L^p(U)$.

A continuous mapping $f\colon X \to Y$ is said to be absolutely continuous on a rectifiable path γ in X if the map $f \circ \gamma_s\colon [0, \mathrm{length}(\gamma)] \to Y$ is absolutely continuous in the usual sense. As in the Euclidean setting, Sobolev maps of metric spaces (which are defined to be continuous) have absolute continuity properties [15, Proposition 3.1].

Theorem 2.1 ([15]). If X is proper, then each $f \in W^{1,p}_{loc}(X;Y)$ is absolutely continuous on p-almost every rectifiable path in X. □

A concept closely related to modulus is capacity. Given disjoint, closed subsets E and F of an open set U in X, we define the condenser $(E,F;U)$ to be the collection of all paths in U, which connect E to F. The p-capacity, $1 \le p < \infty$, of the condenser $(E,F;U)$ is defined by

$$\mathrm{cap}_p(E,F;U) = \inf \int_U \rho^p \, d\mu,$$

where the infimum is taken over all upper gradients ρ of functions $u\colon U \to \mathbb{R}$ such that $u|_E \le 0$ and $u|_F \ge 1$. If we also require that u is Lipschitz, the resulting quantity is called the Lipschitz capacity and denoted by $\mathrm{cap}^L_p(E,F;U)$.

Remark 2.2. Let u be a real-valued function on a metric space X, and set $\widetilde{u} = \min\{u, 1\}$. Then for all $x, y \in X$,

$$|\widetilde{u}(x) - \widetilde{u}(y)| \le |u(x) - u(y)|. \tag{2.2}$$

Hence any upper gradient ρ of u is also an upper gradient of $\widetilde{u}$. Moreover, inequality (2.2) implies that $\widetilde{u}$ is Lipschitz if u is Lipschitz. Thus, in the definitions of capacity and Lipschitz capacity of a condenser $(E,F;U)$, it suffices to only consider functions $u\colon U \to \mathbb{R}$ such that $u|_E \le 0$ and $u|_F = 1$. □

Finding a nontrivial lower bound for the modulus of a path family is frequently difficult. The following theorem [7, Proposition 2.17] provides a tool for doing so.

Theorem 2.3 ([7]). Let (X, d, μ) be a compact and quasiconvex metric measure space, and let E, F be disjoint continua in X. Then for all $q > 0$,

$$\mathrm{cap}^L_q(E,F;X) \le \mathrm{mod}_q(E,F;X).$$

□

Any metric space (X, d) carries a natural family of measures. For any $Q \geq 0$, we define the Q-dimensional Hausdorff measure of a subset $E \subseteq X$ by

$$\mathcal{H}^{Q}_{(X,d)}(E) := \lim_{\epsilon \to 0} \mathcal{H}^{Q,\epsilon}_{(X,d)}(E),$$

where $\mathcal{H}^{Q,\epsilon}_{(X,d)}(E)$ is the Carathéodory premeasure defined as follows. Let $\mathcal{B}_\epsilon$ be the collection of all covers of E by closed sets in X of diameter no greater than ϵ. Then

$$\mathcal{H}^{Q,\epsilon}_{(X,d)}(E) := \inf_{\mathcal{C} \in \mathcal{B}_\epsilon} \sum_{B \in \mathcal{C}} (\operatorname{diam} B)^Q.$$

The Hausdorff dimension of a metric space (X, d) is defined by

$$\dim_H(X) := \sup\{Q \geq 0 : \mathcal{H}^Q(X) > 0\}.$$

For a full description of Hausdorff measure and the Carathéodory construction, see [2, Chapter 2.10]. Note that our definition differs from that in literature as we do not include a dimensional normalization constant.

The Q-dimensional Hausdorff content of a subset $E \subseteq X$ is given by

$$\mathcal{H}^Q_\infty(E) = \inf \sum_{B \in \mathcal{C}} (\operatorname{diam} B)^Q,$$

where the infimum is now taken over all covers $\mathcal{C}$ of E by closed sets in X.

Remark 2.4. It is an easy exercise to show that if $[a, b] \subseteq \mathbb{R}$ is a nondegenerate interval, then for all $Q \geq 0$ we have $\mathcal{H}^Q_\infty([b - a]) = (b - a)^Q$. □

Remark 2.5. If (Y, d) is a metric space, and $E \subseteq X \subseteq Y$, then

$$\mathcal{H}^Q_{(Y,d)}(E) \leq \mathcal{H}^Q_{(X,d)}(E) \leq 2^Q \mathcal{H}^Q_{(Y,d)}(E),$$

and a similar statement holds for Hausdorff content. □

The metric measure space (X, d, μ) is called Ahlfors Q-regular, $Q \geq 0$, if there exists a constant $K \geq 1$ such that for all $a \in X$ and $0 < r \leq \operatorname{diam} X$, we have

$$\frac{r^Q}{K} \leq \mu(\overline{B}_d(a, r)) \leq K r^Q. \tag{2.3}$$

Remark 2.6. If (X, d, μ) is Ahlfors Q-regular with constant K, and $\operatorname{diam} X \leq r \leq C \operatorname{diam} X$ for some $C \geq 1$, then (2.3) remains valid if K is replaced by $C^Q K$. □

Remark 2.7. If (X, d, μ) is Ahlfors Q-regular, then $(X, d, \mathcal{H}^Q)$ is so as well. Thus we may say that a metric space (X, d) is Ahlfors Q-regular if the metric measure space $(X, d, \mathcal{H}^Q)$ is Ahlfors Q-regular. In the sequel, we will deal only with metric measure spaces where the measure is $\mathcal{H}^Q$ for some $Q \geq 0$, eliminating this confusion. □

Remark 2.8. If a metric space (X, d) is Ahlfors Q-regular, then its completion is as well, with a constant depending only on the original constant and Q. See [17, Proposition 2.10]. □

The metric measure space (X, d, μ) is locally Ahlfors Q-regular if for every compact subset $V \subseteq X$, there is a constant $K \geq 1$ and a radius $r_0 > 0$ such that for each point $a \in V$ and radius $0 < r \leq r_0$, the inequalities in (2.3) are satisfied.

The following definition is perhaps nonstandard. Let $E \subseteq X$. We say that (X, d, μ) is locally Ahlfors Q-regular off of E if there is a constant $K \geq 1$ such that for each point $a \in X \backslash E$, there is a radius $r_a > 0$ such that for each $0 < r \leq r_a$, the inequalities in (2.3) are satisfied.

The space (X, d, μ) is said to support a p-Poincaré inequality, $1 \leq p < \infty$ if there are constants $C, \tau \geq 1$ such that if B is a ball in X, $u \colon \tau B \to \mathbb{R}$ is a bounded continuous function, and ρ is an upper gradient of u, then

$$\fint_B |u - u_B| \, d\mu \leq C \operatorname{diam}(B) \left(\fint_{\tau B} \rho^q \, d\mu \right)^{1/q}.$$

Note that if (X, d, μ) supports a p-Poincaré inequality, $1 \leq p < \infty$, then it also supports a q-Poincaré inequality for all $q \geq p$. The Poincaré inequality can be thought of as a requirement that a space contains "many" curves. See [5, 7] for more information.

Many of the properties we have described above are invariant under similarities and bi-Lipschitz maps. We leave the proof of the following proposition to the reader.

Proposition 2.9. Let $(X, d_X, \mathcal{H}^Q)$ and $(Y, d_Y, \mathcal{H}^Q)$ be metric measure spaces, and suppose that $f\colon X \to Y$ is either a similarity or a bi-Lipschitz homeomorphism. Then

(i) if X is quasiconvex, then so is Y,
(ii) if X is Ahlfors Q-regular, then so is Y,
(iii) if X supports a p-Poincaré inequality, then so does Y.

If f is a similarity, then the constant associated with each condition on Y is the same as the constant associated with that condition on X. If f is bi-Lipschitz, then the constant associated with each condition on Y depends only on the constant associated with that condition on X and the bi-Lipschitz constant. □

Metric measure spaces, which are Ahlfors Q-regular and support a p-Poincaré inequality, $p \le Q$, enjoy several important geometric properties [11]. For example, they are quasiconvex with constant depending only on the constants associated with the Ahlfors regularity condition and the Poincaré inequality. Such spaces are also nice analytically. The following theorem, adapted to our needs from the more general [7, Theorem 5.9], shows that capacity type estimates are available.

Theorem 2.10 ([7]). Let $(X, d, \mathcal{H}^Q)$ be a bounded Ahlfors Q-regular metric measure space, which supports a p-Poincaré inequality for some $1 \le p \le Q$. Let E and F be compact subsets of X, and suppose that there are constants $Q - p < s \le Q$ and $\lambda > 0$ such that

$$\min\left\{\mathcal{H}^s_\infty(E), \mathcal{H}^s_\infty(F)\right\} \ge \lambda (\operatorname{diam} X)^s.$$

Then there is a constant $C \ge 1$, depending only on s, λ, and the data associated with X, such that

$$\int_X \rho^p \, d\mathcal{H}^Q \ge C^{-1} (\operatorname{diam} X)^{Q-p},$$

whenever u is a continuous function on X with $u|_E \le a$ and $u|_F \ge b$, where $b - a \ge 1/4$, and ρ is an upper gradient of u. □

3 Cantor Sets

Fix an integer $n \ge 1$. We now discuss the construction of a regular Cantor set C_n of Hausdorff dimension $(\log_3 2)/n$. Let $U_{1,1}$ be an open interval of length $3^{-n}(3^n - 2)$ removed

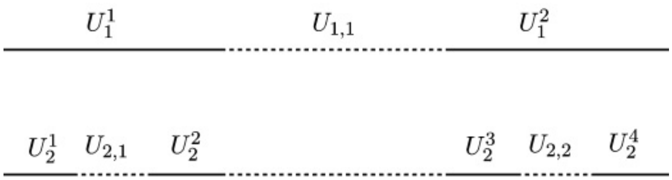

Fig. 1.. The first two steps in the construction of a Cantor set.

from the center of $[0, 1]$, and U_1^1 and U_1^2 the remaining closed intervals of length 3^{-n}. Set

$$\mathcal{C}_n^1 = [0, 1] \backslash U_{1,1} = U_1^1 \cup U_1^2.$$

For $i \geq 1$ an integer, define the index sets

$$J_i := \{1, \dots, 2^{i-1}\} \quad \text{and} \quad K_i := \{1, \dots, 2^i\}.$$

Assume that $\mathcal{C}_n^i$, the intervals $\{U_{i,j}\}_{j \in J_i}$, and the intervals $\{U_i^k\}_{k \in K_i}$ have been defined. For $j \in J_{i+1} = K_i$, define $U_{i+1,j}$ to be the central open subinterval of U_i^j of length $3^{-(i+1)n}(3^n - 2)$, and let $\{U_{i+1}^k\}_{k \in K_{i+1}}$ be the collection of remaining closed subintervals of length $3^{-(i+1)n}$. See Figure 1.

We may now set

$$\mathcal{C}_n^{i+1} = \mathcal{C}_n^i \backslash \bigcup_{j \in J_{i+1}} U_{i+1,j} = \bigcup_{k \in K_{i+1}} U_{i+1}^k.$$

This inductively defines $\mathcal{C}_n^i$ for all positive integers i. Finally, we set

$$\mathcal{C}_n = \bigcap_{i=1}^{\infty} \mathcal{C}_n^i.$$

We may assume that for each positive integer i, the intervals $\{U_i^k\}_{k \in K_i}$ are ordered so that the right endpoint of U_i^k is less than the left endpoint of U_i^{k+1}, and similarly for the intervals $\{U_{i,j}\}_{j \in J_i}$.

For ease of notation, for each positive integer i, we define the positive real number w_i by

$$w_i = \frac{3^{-in}(3^n - 2)}{2},$$

which is half the length of any $U_{i,j}$ where $j \in J_i$. Furthermore, let $u_{i,j}$ be the center of the open interval $U_{i,j}$; thus

$$U_{i,j} = (u_{i,j} - w_i, u_{i,j} + w_i).$$

Similarly, for $k \in K_i$, we set u_i^k to be the center of U_i^k.

The Cantor set $\mathcal{C}_n$ is self-similar in the following sense. For each positive integer i and each $k \in K_i$, there is a 3^{-in}-similarity $\phi_i^k \colon \mathcal{C}_n \to \mathcal{C}_n \cap U_i^k$. Using this, it is not hard to show that for any positive integer i and $k \in K_i$, we have

$$\mathcal{H}^{(\log_3 2)/n}\left(\mathcal{C}_n \cap U_i^k\right) = 2^{-i} = \mathcal{H}_\infty^{(\log_3 2)/n}\left(\mathcal{C}_n \cap U_i^k\right). \tag{3.1}$$

This equation also holds for $i = 0$ under the convention $U_0^1 = [0, 1]$.

Note that the sets $U_{i,j}$ and U_i^k and the quantities w_i depend implicitly on n. Setting $n = 1$ recovers the standard "one-third Cantor set". We will often refer simultaneously to $\mathcal{C}_n$ and $\mathcal{C}_m$ where $m < n$. In this situation we will denote the intervals removed in the construction of $\mathcal{C}_m$ by

$$\{V_{i,j} : i \in \mathbb{Z}^+, j \in J_i\}$$

and the remaining intervals by

$$\left\{V_i^k : i \in \mathbb{Z}^+, k \in K_i\right\}.$$

We will not have need for alternate versions of the quantities w_i defined above; they will always refer to the Cantor set labeled $\mathcal{C}_n$.

Each Cantor set $\mathcal{C}_n$ gives rise to a Cantor function $c_n \colon [0, 1] \to [0, 1]$, which is the unique continuous function satisfying

$$c_n(t) = \frac{2j - 1}{2^i}$$

for $t \in U_{ij}$.

Remark 3.1. The function c_n is not absolutely continuous. To see this, let i be a positive integer and $k \in K_i$, and let $\{I_\alpha\} \subseteq [0, 1]$ be a finite collection of (open or closed) intervals covering $\mathcal{C}_n \cap U_i^k$. Denoting the initial and terminal points of I_α by a_α and b_α, respectively, we have

$$\sum_\alpha |c_n(b_\alpha) - c_n(a_\alpha)| \geq 2^{-i}.$$

This shows that $\mathcal{H}^1(c_n(\mathcal{C}_n \cap U_i^k)) \geq 2^{-i}$, but it follows from the construction that $\mathcal{H}^1(\mathcal{C}_n \cap U_i^k) = 0$. □

We will need a lemma regarding the Hausdorff content of certain sets involving the Cantor sets described above. Because of how it will be used, we employ the notation established for $\mathcal{C}_m$.

Lemma 3.2. Let $i \geq 1$ be an integer. For each $k \in K_i$, suppose we are given sets $E_k, E'_k \subseteq \mathbb{R}$ such that

$$V_i^k \subseteq E_k \cup E'_k.$$

Further suppose that there is a subset $K \subseteq K_i$ with card $K \leq (\text{card } K_i)/2$, such that for all $k \notin K$

$$\mathcal{H}_\infty^{(\log_3 2)/m}(E_k) < \frac{2^{-i}}{4}. \tag{3.2}$$

Then

$$\mathcal{H}_\infty^{(\log_3 2)/m}\left(\bigcup_{k \notin K} E'_k\right) \geq \frac{1}{4}.$$

□

Proof. Towards a contradiction, suppose there is a cover $\{E_\alpha\}_\alpha$ of $\bigcup_{k \notin K} E'_k$ such that

$$\sum_\alpha (\text{diam } E_\alpha)^{(\log_3 2)/m} < \frac{1}{4}.$$

By (3.2), for each $k \notin K$ there is a cover $\{F_\beta^k\}_\beta$ of E_k with

$$\sum_\beta \left(\text{diam } F_\beta^k\right)^{(\log_3 2)/m} \leq \frac{2^{-i}}{4}.$$

Then the collection

$$\{E_\alpha\}_\alpha \cup \bigcup_{k \notin K} \left\{F_\beta^k\right\}_\beta \cup \left\{V_i^k\right\}_{k \in K}$$

covers $\mathcal{C}_m$ and

$$\sum_{\alpha}(\operatorname{diam} E_\alpha)^{(\log_3 2)/m} + \sum_{k\notin K}\sum_{\beta}\left(\operatorname{diam} F^k_\beta\right)^{(\log_3 2)/m} + \sum_{k\in K}\left(\operatorname{diam} V^k_i\right)^{(\log_3 2)/m}$$
$$< \frac{1}{4} + (\operatorname{card} K_i)\frac{2^{-i}}{4} + (\operatorname{card} K)2^{-i} \leq 1.$$

However, by (3.1) we have $\mathcal{H}^{(\log_3 2)/m}_\infty(\mathcal{C}_m) = 1$, yielding a contradiction. ■

4 The Construction and its Basic Properties

In this section we construct the spaces to be used in Theorem 1.5. Fix integers $1 \leq m < n$ such that $n/m \in \mathbb{Z}$. We will consider $\mathcal{C}_n$ and $\mathcal{C}_m$ as defined and notated in the previous section. All subsets of $\mathbb{R}^2$ are endowed with the metric inherited from the standard 2-norm on $\mathbb{R}^2$, which is denoted by $\|\cdot\|$.

For the remainder of the paper, the notation $A \lesssim B$ means that there is a positive constant C depending only on n and m such that $A \leq CB$. The notation $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

For each integer $i \geq 1$, define new index sets J'_i and K'_i by

$$J'_i := \{j \in J_i : U_{i,j} \cap [u_{1,1}, u_{2,2}] \neq \emptyset\},$$
$$K'_i := \left\{k \in K_i : U^k_i \cap [u_{1,1}, u_{2,2}] \neq \emptyset\right\}.$$

We also define

$$X_{1,1} := \{(x, y) \in [0, 1] \times \mathbb{R} : 0 \leq x - u_{1,1} \leq w_1 - \operatorname{dist}(y, \mathcal{C}_m)\},$$
$$X_{2,2} := \{(x, y) \in [0, 1] \times \mathbb{R} : 0 \leq u_{2,2} - x \leq w_2 - \operatorname{dist}(y, \mathcal{C}_m)\}.$$

If $i \geq 3$ is an integer and $j \in J'_i$, define

$$X_{i,j} := \{(x, y) \in [0, 1] \times \mathbb{R} : |x - u_{i,j}| \leq w_i - \operatorname{dist}(y, \mathcal{C}_m)\}. \tag{4.1}$$

Note that the function $y \mapsto \operatorname{dist}(y, \mathcal{C}_m)$ is the maximal 1-Lipschitz function on $\mathbb{R}$, which takes the value 0 at each point of $\mathcal{C}_m$.

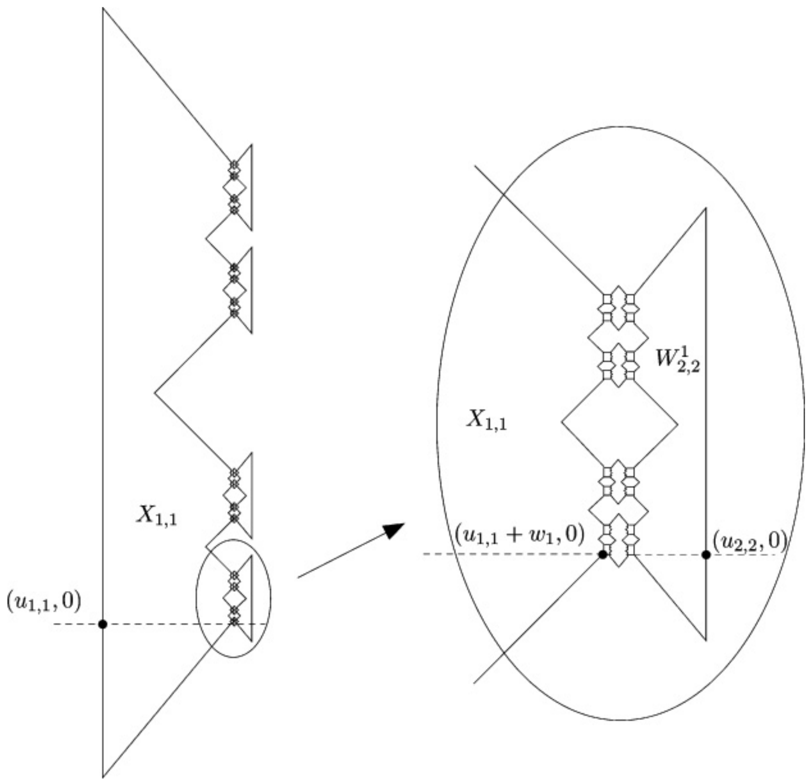

Fig. 2.. At left, X when $n=2$ and $m=1$. At right, a magnified view of a portion of X showing a single component $W^1_{2,2}$ of $X_{2,2}$ and several components of $X_{3,3}$.

We now describe the construction to be used in Theorem 1.5. Set

$$X=\left(\bigcup_{i\geq 1, j\in J'_i} X_{i,j}\right)\cup((\mathcal{C}_n\cap[u_{1,1},u_{2,2}])\times\mathcal{C}_m).$$

See Figure 2. Note that for each $y\in\mathcal{C}_m$, the line segment $[u_{1,1},u_{2,2}]\times\{y\}$ is contained in X.

Define $f:[0,1]\times\mathbb{R}\to[0,1]\times\mathbb{R}$ by

$$f(x,y)=(x,y+c_n(x)). \tag{4.2}$$

Let $Y=f(X)$, and set $E:=X\cap(\mathcal{C}_n\times\mathcal{C}_m)$. We make X and Y into metric measure spaces by equipping them with the ambient metric from $\mathbb{R}^2$ and the measure $\mathcal{H}^2$. By Remark 2.5, $\mathcal{H}^2_X$ is comparable to the restriction of $\mathcal{H}^2_{\mathbb{R}^2}$ to X, and thus they are equivalent for our purposes. A similar statement applies to Y.

Proposition 4.1. The mapping $f\colon X \to Y$ is a homeomorphism and satisfies $H_f(x, y) = 1$ for all points $(x, y) \in X\backslash E$. □

Proof. It is clear that f is a homeomorphism. If $(x, y) \in X\backslash E$, then there are integers $i \geq 1$ and $j \in J_i'$ such that x is an interior point of $U_{i,j}$. It follows that there is an open ball B containing (x, y) such that $f|_B$ is an isometry. The result follows. ■

The remainder of this section is devoted to establishing geometric and analytic properties of X and Y. To do so, we first examine the sets $X_{i,j}$. For a given $y \in \mathbb{R}$, the inequality defining $X_{i,j}$ has a solution $x \in [0, 1]$ if and only if $y \in \overline{\mathcal{N}}_{w_i}(\mathcal{C}_m)$. Consequently, the following lemma will enable us to describe the components of $X_{i,j}$.

Lemma 4.2. Let $i \geq 1$ be an integer. Then

$$\overline{\mathcal{N}}_{w_i}(\mathcal{C}_m) = \bigcup_{k\in K_{(i-1)n/m}} \overline{\mathcal{N}}_{w_i}\left(V^k_{(i-1)n/m}\right), \tag{4.3}$$

and the union is disjoint. □

Proof. It follows from the definitions that

$$\overline{\mathcal{N}}_{w_i}(\mathcal{C}_m) \subseteq \bigcup_{k\in K_{(i-1)n/m}} \overline{\mathcal{N}}_{w_i}\left(V^k_{(i-1)n/m}\right). \tag{4.4}$$

On the other hand, suppose $y \in \overline{\mathcal{N}}_{w_i}(V^k_{(i-1)n/m})$ for some $k \in K_{(i-1)n/m}$. If $y \notin V^k_{(i-1)n/m}$, then y is a distance at most w_i from one of the endpoints of $V^k_{(i-1)n/m}$, which are in $\mathcal{C}_m$. If $y \in V^k_{(i-1)n/m}$, we note that either $y \in \mathcal{C}_m$ or there exists an integer $i_0 > (i-1)n/m$ and an integer $j_0 \in J_{i_0}$ such that $y \in V_{i_0,j_0}$, the endpoints of which are contained in $\mathcal{C}_m$. We have

$$|V_{i_0,j_0}| \leq 3^{-(\frac{(i-1)n}{m}+1)m}(3^m - 2) = 3^{-in}3^{n-m}(3^m - 2) \leq 2w_i.$$

Thus $\operatorname{dist}(y, \mathcal{C}) \leq w_i$. This, along with (4.4), shows that (4.3) holds. To see that the union is disjoint, consider that any two sets $V^k_{(i-1)n/m}$ and $V^{k'}_{(i-1)n/m}$ with $k \neq k'$ are separated by the interval $V_{(i-1)n/m,j}$ for some $j \in J_{(i-1)n/m}$. Disjointness now follows from

$$|V_{(i-1)n/m,j}| = 3^{-in}3^n(3^m - 2) > 2w_i,$$

completing the proof. ■

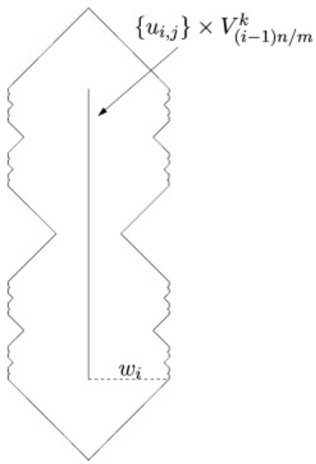

Fig. 3.. A component $W^k_{i,j}$ of $X^k_{i,j}$ when $i \geq 3$

If points (x_1, y) and (x_2, y) are in some $X_{i,j}$, then the horizontal line segment $[x_1, x_2] \times y$ is also contained in $X_{i,j}$. Thus Lemma 4.2 implies that the components of $X_{i,j}$ may be indexed by $k \in K_{(i-1)n/m}$; we denote them by

$$W^k_{i,j} = \{(x, y) \in [0, 1] \times \overline{\mathcal{N}}_{w_i}\left(V^k_{(i-1)n/m}\right) : |x - u_{i,j}| \leq w_i - \text{dist}(y, \mathcal{C}_m)\}, \tag{4.5}$$

with the obvious modifications for the components of $X_{1,1}$ and $X_{2,2}$. See Figures 2 and 3.

The following two remarks follow immediately from this description. Let $i \geq 1$ be an integer, $j \in J'_i$, and $k, k' \in K_{(i-1)n/m}$.

Remark 4.3. If $(x, y) \in W^k_{i,j}$ and $(x', y') \in W^{k'}_{i,j}$, where $k \neq k'$, then

$$|y' - y| \geq \text{dist}\left(V^k_{(i-1)n/m}, V^{k'}_{(i-1)n/m}\right) - 2w_i \geq 3^{-in}(3^{n+m} - 3^{n+1} + 2) \geq 2(3^{-in}).$$

□

Remark 4.4. If $(x, y) \in W^k_{i,j}$, then there is a point $y' \in \mathcal{C}_m$ such that $(x, y') \in W^k_{i,j}$ and

$$|y - y'| \leq \text{dist}(x, \partial U_{i,j}).$$

Moreover, y' may be chosen so that the line segment connecting (x, y) to (x, y') is contained in $W^k_{i,j}$. □

The following elementary but useful lemma follows from the fact that in the construction of $\mathcal{C}_n$, each removed interval is comparable in length to each interval not removed.

Lemma 4.5. Let $i > 1$ be an integer, $j \in J'_i$, and $i_0 < i$. Then there is an index $j_0 \in J'_{i_0}$ such that for any $x \in U_{i,j}$,

$$|x - u_{i_0,j_0}| \le 2w_{i_0}.$$

□

Proof. Since $i_0 < i$ and $j \in J'_i$, the construction of $\mathcal{C}_n$ implies that there is some $k_0 \in K'_{i_0}$ such that $U_{i,j} \subseteq U^{k_0}_{i_0}$. We may find $j_0 \in J'_{i_0}$ such that $U^{k_0}_{i_0}$ is adjacent to U_{i_0,j_0}. Now

$$|x - u_{i_0,j_0}| \le \left|U^{k_0}_{i_0}\right| + w_{i_0} \le 3^{-i_0 n} + w_{i_0} \le 2w_{i_0}.$$

■

The next proposition describes the self-similarity of the space X. Let

$$Z = \{(x, y) \in [0, 1] \times \mathbb{R} : |x - u_{1,1}| \le w_1 - \mathrm{dist}(y, \mathcal{C}_m)\}.$$

Note that

$$X_{1,1} = Z \cap ([u_{1,1}, u_{1,1} + w_1] \times \mathbb{R}).$$

Proposition 4.6. The set $X_{1,1}$ is connected and hence equal to $W^1_{1,1}$. For each $k \in K_{n/m}$, there is a 3^{-n}-similarity $s^k_{2,2}\colon X_{1,1} \to W^k_{2,2}$. For all integers $i \ge 3$, $j \in J'_i$, and $k \in K_{(i-1)n/m}$, there is a $3^{-(i-1)n}$-similarity $s^k_{i,j}\colon Z \to W^k_{i,j}$. In this case, $s^k_{i,j}$ maps $X_{1,1}$ onto $W^k_{i,j} \cap ([u_{i,j}, u_{i,j} + w_i] \times \mathbb{R})$. □

Proof. The first assertion is clear. The second is proven similarly to the third, which we now establish. Fix integers $i \ge 3$, $j \in J'_i$ and $k \in K_{(i-1)n/m}$. Define $s^k_i : \mathbb{R} \to \mathbb{R}$ by

$$s^k_i(y) = \frac{(y - 1/2)}{3^{(i-1)n}} + v^k_{(i-1)n/m}.$$

Then s^k_i maps $\mathcal{C}$ bijectively onto $\mathcal{C} \cap V^k_{(i-1)n/m}$. Furthermore,

$$\mathrm{dist}\left(s^k_i(y), \mathcal{C}_m\right) = \frac{\mathrm{dist}(y, \mathcal{C}_m)}{3^{(i-1)n}}.$$

Set

$$s^k_{i,j}(x, y) = \left(\frac{w_i}{w_1}(x - u_{1,1}) + u_{i,j}, s^k_i(y)\right).$$

Noting that $w_i/w_1 = 3^{-(i-1)n}$, (4.1) and (4.5) show that $s^k_{i,j}$ is a homeomorphism from Z to $W^k_{i,j}$; a simple calculation shows that it is a $3^{-(i-1)n}$-similarity.

The final assertion is easily verified. ■

The following proposition is tedious but elementary to verify. We omit the proof.

Proposition 4.7. Let $i \geq 2$ be an integer. There is a constant $L \geq 1$, independent of n, m, and i, such that each of $X_{1,1}$, Z, and $X_{1,1} \cap ([u_{1,1}, u_{1,1} + (w_1 - w_i)])$ are L-bi-Lipschitz equivalent to the unit square $[0, 1]^2$. □

Propositions 2.9, 4.6, and 4.7, along with the fact that the unit square is quasiconvex, Ahlfors 2-regular, and supports a 1-Poincaré inequality, show the following corollary.

Corollary 4.8. Let $i \geq 2$, $j \in J_i'$, and $k \in K_{(i-1)n}$. The sets $X_{1,1}$, $W_{i,j}^k$,

$$X_{1,1} \cap ([u_{1,1}, u_{1,1} + (w_1 - w_i)]), \quad \text{and} \quad W_{i,j}^k \cap ([u_{i,j}, u_{i,j} + w_i] \times \mathbb{R})$$

are quasiconvex, Ahlfors 2-regular, and support a 1-Poincaré inequality. Moreover, the constants associated with each condition are independent of i, j, k, n, and m. □

We now establish some global properties of X and Y. It is clear that both spaces are compact.

Remark 4.9. As the collection $\{\partial U_{i,j}\}_{i \in \mathbb{Z}^+, j \in J_i}$ is dense in $\mathcal{C}_m \cap [u_{1,1}, u_{2,2}]$, we have

$$X = \overline{\bigcup_{i \in \mathbb{Z}^+, j \in J_i'} X_{i,j}}.$$

□

Proposition 4.10. The space X is Λ-quasiconvex where Λ depends only on n and m. □

Proof. Let (x_1, y_1) and (x_2, y_2) be points in X. We wish to show that there is a path γ in X connecting these points such that

$$\text{length}(\gamma) \lesssim \|(x_1, y_1) - (x_2, y_2)\|. \tag{4.6}$$

Remark 4.9 shows that we may assume

$$(x_1, y_1), (x_2, y_2) \in \bigcup_{i \in \mathbb{Z}^+, j \in J_i'} X_{i,j}.$$

Case 1. Assume that there is some integer $i \geq 1$ and $j \in J_i'$ such that both (x_1, y_1) and (x_2, y_2) are in $X_{i,j}$. Let $(x_1, y_1) \in W_{i,j}^{k_1}$ and $(x_2, y_2) \in W_{i,j}^{k_2}$ where $k_1, k_2 \in K_{(i-1)n/m}$.

If $|y_1 - y_2| < 2(3^{-in})$, then Remark 4.3 implies that $k_1 = k_2$, and the desired path connecting (x_1, y_1) to (x_2, y_2) exists by Corollary 4.8. Thus we may assume that there is some integer $i_0 < i$ such that

$$2\left(3^{-(i_0+1)n}\right) \leq |y_1 - y_2| < 2(3^{-i_0 n}), \tag{4.7}$$

Assume for the moment that y_1 and y_2 are points in $\mathcal{C}_m$. By Lemma 4.5, there is an index $j_0 \in J_{i_0}'$ such that

$$\max\{|x_1 - u_{i_0,j_0}|, |x_2 - u_{i_0,j_0}|\} \leq 2w_{i_0}.$$

Let β_1 be a path parameterizing the horizontal line segment from (x_1, y_1) to (u_{i_0,j_0}, y_1), and let β_2 be a path parameterizing the horizontal line segment from (u_{i_0,j_0}, y_2) to (x_2, y_2). Since $y_1, y_2 \in \mathcal{C}_m$, these paths are in X. The second inequality in (4.7) along with Remark 4.3 implies that there is some $k_0 \in K_{(i_0-1)n/m}$ such that (u_{i_0,j_0}, y_1) and (u_{i_0,j_0}, y_2) are in $W_{i_0,j_0}^{k_0}$. By Corollary 4.8, there is a path γ_0 connecting these points with

$$\text{length}(\gamma_0) \lesssim |y_1 - y_2|.$$

Thus the concantenation $\gamma_1 = \beta_1 \cdot \gamma_0 \cdot \beta_2$ is a path in X connecting (x_1, y_1) to (x_2, y_2) with

$$\text{length}(\gamma_1) \lesssim |y_1 - y_2| + 4w_{i_0} \lesssim |y_1 - y_2| \lesssim \|(x_1, y_1) - (x_2, y_2)\|. \tag{4.8}$$

We now remove the assumption that $y_1, y_2 \in \mathcal{C}_m$. By Remark 4.4, there is a point $y_1' \in \mathcal{C}_m$ and a path α_1 in $W_{i,j}^{k_1}$ parameterizing the line segment from (x_1, y_1) to (x_1, y_1') with $\text{length}(\alpha_1) \leq w_i$. Similarly, there is a point $y_2' \in \mathcal{C}_m$ and a path α_2 in $W_{i,j}^{k_2}$ parameterizing the line segment from (x_2, y_2') to (x_2, y_2) with $\text{length}(\alpha_2) \leq w_i$. From the first inequality in (4.7), we see that $w_i \lesssim |y_1 - y_2|$. Thus

$$\|(x_1, y_1') - (x_2, y_2')\| \leq \|(x_1, y_1) - (x_2, y_2)\| + 2w_i \lesssim \|(x_1, y_1) - (x_2, y_2)\|.$$

By the discussion leading to (4.8), there is a path γ_1 connecting (x_1, y_1') to (x_2, y_2') with

$$\text{length}(\gamma_1) \lesssim \|(x_1, y_1') - (x_2, y_2')\| \lesssim \|(x_1, y_1) - (x_2, y_2)\|.$$

Thus $\gamma = \alpha_1 \cdot \gamma_1 \cdot \alpha_2$ is a path in X connecting (x_1, y_1) to (x_2, y_2) with

$$\text{length}(\gamma) \lesssim w_i + \|(x_1, y_1) - (x_2, y_2)\| + w_i \lesssim \|(x_1, y_1) - (x_2, y_2)\|.$$

Case 2. Now assume that there are integers $i_1, i_2 \geq 1$, $j_1 \in J'_{i_1}$, $j_2 \in J'_{i_2}$, $k_1 \in K_{(i_1-1)n/m}$, and $k_2 \in K_{(i_2-1)n/m}$, such that $(x_1, y_1) \in W^{k_1}_{i_1,j_1}$ and $(x_2, y_2) \in W^{k_2}_{i_2,j_2}$, where $X_{i_1,j_1} \neq X_{i_2,j_2}$. This implies that

$$\text{dist}(x_1, \partial U_{i_1,j_1}) \leq |x_1 - x_2|.$$

Thus by Remark 4.4, we may find a point $y'_1 \in \mathcal{C}_m$ such that $(x_1, y'_1) \in W^{k_1}_{i_1,j_1}$, and there is a path α in X parameterizing the line segment connecting (x_1, y_1) to (x_1, y'_1) with

$$\text{length}(\alpha) = |y_1 - y'_1| \leq \text{dist}(x_1, \partial U_{i_1,j_1}) \leq |x_1 - x_2|. \tag{4.9}$$

Since $y'_1 \in \mathcal{C}_m$, there is a path β in X parameterizing the line segment connecting (x_1, y'_1) to (x_2, y'_1) with

$$\text{length}(\beta) = |x_1 - x_2|.$$

Consider that by (4.9),

$$\|(x_2, y'_1) - (x_2, y_2)\| \leq |y_1 - y'_1| + |y_1 - y_2| \lesssim \|(x_1, y_1) - (x_2, y_2)\|.$$

Noting that (x_2, y'_1) and (x_2, y_2) are both in X_{i_2,j_2}, Case 1 above provides a path γ connecting them with

$$\text{length}(\gamma) \lesssim \|(x_1, y_1) - (x_2, y_2)\|.$$

Now $\alpha \cdot \beta \cdot \gamma$ is a path in X connecting (x_1, y_1) to (x_2, y_2) with

$$\text{length}(\alpha \cdot \beta \cdot \gamma) \lesssim \|(x_1, y_1) - (x_2, y_2)\|,$$

as desired.

These cases exhaust all possibilities, completing the proof. ∎

Proposition 4.11. The space X is Ahlfors 2-regular, with constant depending only on n and m. □

Proof. Let $(x, y) \in X$, and $r \leq \operatorname{diam}(X) \lesssim 1$. By Remarks 2.8 and 4.9, we may assume that there are indices $i \in \mathbb{Z}$, $j \in J_i'$, and $k \in K_{(i-1)n/m}$ such that $(x, y) \in W_{i,j}^k$.

Since X is endowed with the ambient metric from $\mathbb{R}^2$, it follows from Remark 2.5 and the Ahlfors 2-regularity of the plane that there is a constant $\kappa \geq 1$, not even depending on n, such that

$$\mathcal{H}_X^2(\overline{B}_X((x, y), r)) \leq \kappa r^2.$$

Thus it suffices to show the corresponding lower bound.

If $r \leq 6(3^n)w_i$, then by Proposition 4.6 we see that $r \lesssim \operatorname{diam} W_{i,j}^k$. Hence by Corollary 4.8 and Remark 2.6, there is a constant K depending only on n such that

$$\mathcal{H}_X^2(\overline{B}_X((x, y), r)) \gtrsim r^2 \tag{4.10}$$

If $6(3^n)w_i \leq r \leq \operatorname{diam} X$, then we may find an integer $1 \leq i_0 < i$ such that

$$6w_{i_0} \leq r \leq 6(3^n)w_{i_0}.$$

Note that $6(3^n)w_1 \geq 6 \geq \operatorname{diam} X$, so this case is not possible for $i = 1$. Combining Remark 4.4 and Lemma 4.5, we may find indices $j_0 \in J_{i_0}'$ and $k_0 \in K_{(i_0-1)n/m}$ and a point $(x', y') \in W_{i_0,j_0}^{k_0}$ and

$$\|(x, y) - (x', y')\| \leq 3w_{i_0} \leq r/2.$$

Thus

$$\overline{B}_X((x, y), r) \supseteq \overline{B}_X((x', y'), r/2).$$

and hence the discussion leading to (4.10) implies

$$\mathcal{H}_X^2(\overline{B}_X((x, y), r)) \geq \mathcal{H}_X^2(\overline{B}_X((x', y'), r/2)) \gtrsim r^2.$$

Thus X is Ahlfors 2-regular with constant depending only on n and m. ■

Proposition 4.12. The space Y is locally Ahlfors 2-regular off $f(E)$. □

Proof. This follows from Corollary 4.8 and the fact that f is an isometry when restricted to the interior of each $X_{i,j}$. ■

5 The Proof of Theorem 1.5

For integers $1 \leq m < n$ with $n/m \in \mathbb{Z}$, let X, E, Y, and f be as defined in Section 4. For the remainder of the paper we will use the following notation. If L is a vertical line in $\mathbb{R}^2$ and $(x, y) \in X$, denote by $\overline{(x, y)}_L$ the reflection of the point (x, y) across the line L. Given vertical lines L and L' that intersect the x-axis in the points l and l', respectively, set

$$[L_1, L_2] = X \cap ([l, l'] \times \mathbb{R}).$$

Let L_0 be the vertical line $\{u_{1,1}\} \times \mathbb{R}$ and R_0 be the vertical line $\{u_{2,2}\} \times \mathbb{R}$. Define the path families

$$\Gamma = \{\gamma : [0, 1] \to X : \gamma(0) \in L_0, \gamma(1) \in R_0\},$$
$$\Gamma_0 = \{\gamma \in \Gamma : f \circ \gamma_s \text{ is absolutely continuous}\}.$$

We now show that the curve family Γ_0 may be ignored in computing the modulus of Γ.

If $\gamma : I \to \mathbb{R}^2$ is any path, we denote the components of γ by γ^x and γ^y, so that for any $t \in I$,

$$\gamma(t) = (\gamma^x(t), \gamma^y(t)).$$

Lemma 5.1. If $\gamma \in \Gamma_0$, then $\mathcal{H}^1(((\gamma_s)^x)^{-1}(\mathcal{C}_n)) > 0$. □

Proof. It suffices to show that if $\mathcal{H}^1(((\gamma_s)^x)^{-1}(\mathcal{C}_n)) = 0$, then $f \circ \gamma_s$ is not absolutely continuous. Set $F = ((\gamma_s)^x)^{-1}(\mathcal{C}_n)$, and let $0 < \delta < 1/8$. By assumption we may find finitely many intervals $\{(a_\alpha, b_\alpha)\} \subseteq [0, \text{length}(\gamma)]$ which cover F and satisfy

$$\sum_\alpha |b_\alpha - a_\alpha| < \delta.$$

Since γ_s is continuous and connects L_0 to R_0, we have that

$$\bigcup_\alpha \left(\gamma_s^x(a_\alpha, b_\alpha)\right) \supseteq \mathcal{C}_n \cap U_2^3.$$

The fact that the 1-norm is equivalent to the 2-norm on $\mathbb{R}^2$, the definition of f, and the triangle inequality show that there is a universal constant $c > 0$ such that

$$\sum_\alpha \|f \circ \gamma_s(b_\alpha) - f \circ \gamma_s(a_\alpha)\| \geq c \sum_\alpha \left|c_n \circ \gamma_s^x(b_\alpha) - c_n \circ \gamma_s^x(a_\alpha)\right| - \left|\gamma_s^y(b_\alpha) - \gamma_s^y(a_\alpha)\right|.$$

From Remark 3.1 and the fact that γ_s^y is 1-Lipschitz, we may conclude that

$$\sum_\alpha \|f \circ \gamma_s(b_\alpha) - f \circ \gamma_s(a_\alpha)\| \geq c(1/4 - \delta) \geq c/8.$$

Since δ can be made arbitrarily small, this shows that $f \circ \gamma_s$ is not absolutely continuous. ■

Lemma 5.2. Let $q > 0$. Then $\mathrm{mod}_q(\Gamma) = \mathrm{mod}_q(\Gamma \backslash \Gamma_0)$. □

Proof. It follows from the definitions that $\mathrm{mod}_q(\Gamma \backslash \Gamma_0) \leq \mathrm{mod}_q(\Gamma)$. Thus it suffices to show that $\mathrm{mod}_q(\Gamma \backslash \Gamma_0) \geq \mathrm{mod}_q(\Gamma)$. Let $\rho\colon X \to [0, \infty]$ be admissible for $\Gamma \backslash \Gamma_0$, and define $\rho'\colon X \to [0, \infty]$ by

$$\rho'(x, y) = \begin{cases} \infty & x \in \mathcal{C}_n \\ \rho(x, y) & x \notin \mathcal{C}_n. \end{cases}$$

Then ρ' is a Borel function, and it follows from Lemma 5.1 that ρ' is admissible for Γ. Since $\mathcal{H}^2((\mathcal{C}_n \times \mathbb{R}) \cap X) = 0$, we have

$$\int_X \rho'^q \, d\mathcal{H}^2 = \int_X \rho^q \, d\mathcal{H}^2.$$

Taking the infimum over all functions ρ that are admissible for $\Gamma \backslash \Gamma_0$ yields the desired result. ■

Proof of Theorem 1.5. We begin by determining the correct parameters n and m to use in the construction, and setting the value of q in condition (v).

Let $\alpha > 0$ and $\epsilon > 0$ be given. Fix a positive integer m such that $(2\log_3 2)/m \leq \min\{\alpha, \epsilon\}$. For any positive integer n, we define

$$q(n) = 2 - (\log_3 2)\left(\frac{1}{n} + \frac{1}{m}\right) + n^{-1/2} \quad \text{and} \quad \delta(n) = \frac{q(n)(q(n)-1)\log_3 2}{n + (q(n) - 1 - n)\log_3 2}. \tag{5.1}$$

Note that if $n^{-1/2} \leq \log_3 2/m$, then

$$1 < q(n) < 2 \quad \text{and} \quad 0 < \delta(n) < \frac{2\log_3 2}{n(1-\log_3 2)}. \tag{5.2}$$

We now fix an integer $n > m$ that is a multiple of m, and so large that $n^{-1/2} \leq \log_3 2/m$ and the following inequalities are satisfied:

$$(\log_3 2)\left(\frac{1}{n} + \frac{1}{m}\right) - \frac{((n/m)-1)\log_3 2}{n - \log_3 2} \leq n^{-1/2}, \tag{5.3}$$

$$\delta(n) + \frac{\log_3 2}{n} \leq n^{-1/2}. \tag{5.4}$$

This is possible because for sufficiently large n, the quantities on the left-hand sides of (5.3) and (5.4) are bounded above by a linear function of $1/n$. Set $q = q(n)$, $\delta = \delta(n)$. Inequalities (5.3) and (5.4) now imply the following:

$$1 < \frac{\left(2 - \frac{\log_3 2}{m}\right)n - \log_3 2}{n - \log_3 2} \leq q \tag{5.5}$$

$$0 < \delta \leq \frac{\log_3 2}{m} - (2-q). \tag{5.6}$$

Let X, Y, E, and f be as defined in Section 4 with parameters n and m as above. Then, as $(2\log_3 2)/m \leq \alpha$ and $n > m$, we have

$$\dim_H(E) = (\log_3 2)\left(\frac{1}{n} + \frac{1}{m}\right) \leq \alpha \quad \text{and} \quad 0 < \mathcal{H}^{\dim_H(E)}(E) < \infty.$$

These facts, along with Propositions 4.1, 4.10, 4.11, and 4.12, provide the information needed to show that X, Y, f, and E satisfy conditions (i)–(iv) of Theorem 1.5. As $n^{-1/2} \leq \log_3 2/m < \epsilon$, we see that $q < 2 - \dim_H(E) + \epsilon$. It remains to show that $f \notin W^{1,q}_{loc}(X;Y)$. Since f is not absolutely continuous on any path in the family $\Gamma\backslash\Gamma_0$ defined above, by Theorem 2.1 we need only show that $\mathrm{mod}_q(\Gamma\backslash\Gamma_0) > 0$.

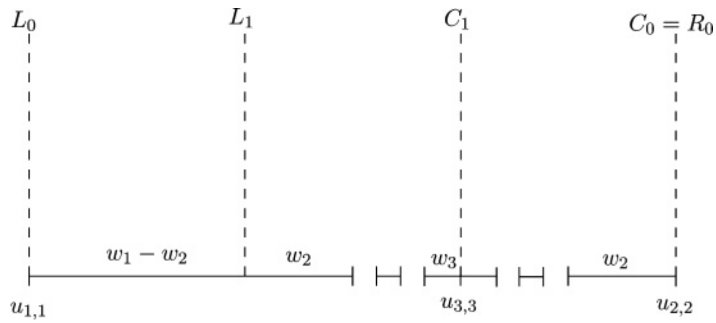

Fig. 4.. The first few lines.

By Theorem 2.3 and Lemma 5.2, we have

$$\mathrm{cap}_q^L(L_0, R_0; X) \le \mathrm{mod}_q(\Gamma) = \mathrm{mod}_q(\Gamma \backslash \Gamma_0).$$

Thus it suffices to show that $\mathrm{cap}_q^L(L_0, R_0; X) > 0$. As per Remark 2.2, let $v : X \to \mathbb{R}$ be a c-Lipschitz function, $c \ge 1$, with $v|_{L_0} \le 0$ and $v|_{R_0} = 1$. Let $\rho\colon X \to [0, \infty]$ be an upper gradient of v in X. We will show that

$$\int_X \rho^q \, d\mathcal{H}^2 \gtrsim \kappa > 0, \tag{5.7}$$

where $\kappa > 0$ depends only on n and m. The basic idea is to use the symmetry and self-similarity of the space X to keep track of where the function v grows.

We now inductively define some sequences of lines and functions. Set $C_0 = R_0$, $v_0 = v$, and $\rho_0 = \rho$. Let L_1 be the vertical line through the point $(u_{1,1} + (w_1 - w_2), 0)$. Notice that

$$u_{3,3} = \frac{u_{1,1} + (w_1 - w_2) + u_{2,2}}{2}.$$

Let C_1 be the vertical line through $(u_{3,3}, 0)$. Then the reflection of the line L_1 through the line C_1 is C_0. See Figure 4.

Define $v_1 : [L_1, C_1] \to \mathbb{R}$ by

$$v_1(x, y) = v_0(x, y) + 1 - v_0(\overline{(x, y)}_{C_1}).$$

Then v_1 is $2c$-Lipschitz,

$$v_1|_{L_1} = v_0|_{L_1} + 1 - v_0|_{R_0} = v_0|_{L_1},$$

and $v_1|_{C_1} = 1$. Furthermore, the function $\rho_1\colon [L_1, C_1] \to [0,\infty]$ defined by

$$\rho_1(x,y) = \rho_0(x,y) + \rho_0(\overline{(x,y)}_{C_1})$$

is an upper gradient of v_1. We have

$$\int_X \rho_0^q \, d\mathcal{H}^2 \geq \int_{[L_0,L_1]} \rho_0^q \, d\mathcal{H}^2 + 2^{(1-q)} \int_{[L_1,C_1]} \rho_1^q \, d\mathcal{H}^2.$$

Now, let $i \geq 2$ and assume that $L_{i-1}, C_{i-1}, v_{i-1}$, and ρ_{i-1} are defined. Let L_i be the vertical line through the point $(u_{1,1} + (w_1 - w_{i+1}), 0)$, and let C_i be the vertical line such that the reflection of L_i in C_i is C_{i-1}. Define $v_i : [L_i, C_i] \to \mathbb{R}$ by

$$v_i(x,y) = v_{i-1}(x,y) + 1 - v_{i-1}(\overline{(x,y)}_{C_i}).$$

We see inductively that v_i is $2^i c$ Lipschitz and that $v_i|_{L_i} = v_{i-1}|_{L_i}$ and $v_i|_{C_i} = 1$. Furthermore, the function $\rho_i : [L_i, C_i] \to [0,\infty]$ defined by

$$\rho_i(x,y) = \rho_{i-1}(x,y) + \rho_{i-1}(\overline{(x,y)}_{C_i})$$

is an upper gradient of v_i. Finally, we see by induction that for any positive integer i_0, the following inequality holds:

$$\int_X \rho^q \, d\mathcal{H}^2 \geq \sum_{i=0}^{i_0-1} 2^{(1-q)i} \int_{[L_i,L_{i+1}]} \rho_i^q \, d\mathcal{H}^2 + 2^{(1-q)i_0} \int_{[L_{i_0},C_{i_0}]} \rho_{i_0}^q \, d\mathcal{H}^2. \tag{5.8}$$

We will use the first term on the right-hand side of inequality (5.8) to keep track of the change of v on large pieces of X, and the second term for small pieces.

Let M be the vertical line $\{u_{1,1} + w_1\} \times \mathbb{R}$. If $(x,y) \in X \cap M$, then $y \in \mathcal{C}_m$ and

$$\operatorname{dist}((x,y), X \cap C_i) = 3^{-(i+2)n} + w_{i+2} = \frac{3^{-(i+1)n}}{2}$$

for each integer $i \geq 1$. Since $v_i|_{C_i} = 1$ and v_i is $2^i c$-Lipschitz, we may find an integer $i_0 \geq 2$ such that $v_{i_0}|_M \geq 3/4$. Note that i_0 depends on c, and so our final estimates should be independent of i_0. See Figure 5.

The line C_{i_0-1} intersects the x-axis at the point u_{i_0+1,j_0} for where $j_0 = 2^{i_0-1} + 1 \in J'_{i_0+1}$. By the symmetry of $[L_{i_0}, C_{i_0-1}]$ about the vertical line C_{i_0}, and the symmetry of

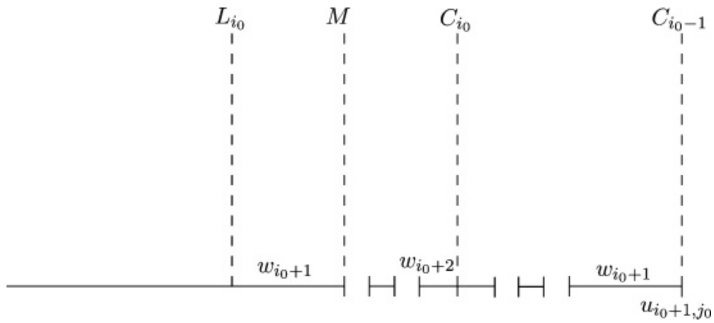

Fig. 5.. After choosing i_0.

X_{i_0+1,j_0} about the line C_{i_0-1}, we see that there is an isometry from $[L_{i_0}, M]$ to

$$X_{i_0+1,j_0} \cap ([u_{i_0+1,j_0}, u_{i_0+1,j_0} + w_{i_0+1}] \times \mathbb{R}).$$

Thus we may index the components of $[L_{i_0}, M]$ by $K_{i_0 n/m}$. For each $k \in K_{i_0 n/m}$, set W_k to be the component of $[L_{i_0}, M]$ that corresponds to

$$W^k_{i_0+1,j_0} \cap ([u_{i_0+1,j_0}, u_{i_0+1,j_0} + w_{i_0+1}] \times \mathbb{R}).$$

Note that by Corollary 4.8, the components W_k of $[L_{i_0}, M]$ are Ahlfors 2-regular and support a 1-Poincaré inequality (and hence a q-Poincaré inequality).

For each $k \in K_{i_0 n/m}$, set

$$E_k = \{(x, y) \in W_k \cap L_{i_0} : v_{i_0}(x, y) < 1/2\}.$$

Note that $W_k \cap L_{i_0}$ is equal to the vertical line segment

$$\{u_{1,1} + (w_1 - w_{i_0+1})\} \times \overline{\mathcal{N}}_{w_{i_0+1}}\left(V^k_{i_0 n/m}\right).$$

Thus by Remarks 2.4 and 2.5, we see that

$$\mathcal{H}^{(\log_3 2)/m}_\infty (W_k \cap L_{i_0}) \approx 2^{-i_0 n/m}.$$

We now define a collection of indices that selects the components W_k on which the value of v_{i_0} on L_{i_0} is "often" less than $1/2$, in the sense of the appropriate Hausdorff content:

$$K_S := \left\{ k \in K_{i_0 n/m} : \mathcal{H}_\infty^{(\log_3 2)/m}(E_k) \geq \frac{2^{-i_0 n/m}}{4} \right\}. \tag{5.9}$$

Informally, an index $k \in K_{i_0 n/m}$ is in K_S if v changes substantially on small pieces of X that are roughly at the height of $V^k_{i_0 n/m}$.

It follows from Proposition 4.6 that

$$(\operatorname{diam} W_k)^{(\log_3 2)/m} = (3^{-i_0 n} \operatorname{diam} X_{1,1})^{(\log_3 2)/m} \approx 2^{-i_0 n/m}.$$

Thus for each $k \in K_S$,

$$\mathcal{H}_\infty^{(\log_3 2)/m}(E_k) \gtrsim (\operatorname{diam} W_k)^{(\log_3 2)/m}.$$

For $k \in K_S$, set

$$F_k = W_k \cap M = \{u_{1,1} + w_1\} \times \left(\mathcal{C}_m \cap V^k_{i_0 n/m}\right).$$

Then $v_{i_0} \geq 3/4$ on F_k. Moreover, Remark 2.5 and Equation (3.1) imply that

$$\mathcal{H}_\infty^{(\log_3 2)/m}(F_k) \approx \mathcal{H}_\infty^{(\log_3 2)/m}\left(\mathcal{C}_m \cap V^k_{i_0 n/m}\right) = 2^{-i_0 n/m} \approx (\operatorname{diam} W_k)^{(\log_3 2)/m}.$$

Note that (5.6) implies $2 - q < (\log_3 2)/m \leq 2$. Furthermore, we have $1 \leq q \leq 2$. Thus for $k \in K_S$, we may apply Theorem 2.10 to the space W_k, using the sets E_k and F_k, and the function v_{i_0}. The conclusion is that for each $k \in K_S$,

$$\int_{W_k} \rho_{i_0}^q \, d\mathcal{H}^2 \gtrsim (\operatorname{diam} W_k)^{2-q} \gtrsim 3^{-i_0 n(2-q)}.$$

We now proceed in two cases. First suppose that $\operatorname{card}(K_S) \geq (\operatorname{card} K_{i_0 n/m})/2 = 2^{i_0 n/m - 1}$. As $\{W_k\}_{k \in K_S}$ is a disjointed family of measurable subsets of $[L_{i_0}, C_{i_0}]$, we have

$$2^{(1-q)i_0} \int_{[L_{i_0}, C_{i_0}]} \rho_{i_0}^q \geq 2^{(1-q)i_0} \left(\sum_{k \in K_S} \int_{W_k} \rho_{i_0}^q \, d\mathcal{H}^2 \right) \gtrsim 2^{(1-q)i_0} 2^{i_0 n/m} 3^{-i_0 n(2-q)}.$$

The final inequality in (5.5) is equivalent to

$$i_0\left((1-q)+\frac{n}{m}-\frac{n(2-q)}{\log_3 2}\right)\geq 0,$$

and so

$$2^{(1-q)i_0}\int_{[L_{i_0},C_{i_0}]}\rho_{i_0}^q \gtrsim 1.$$

By inequality (5.8), this yields the desired conclusion (5.7).

Now suppose that card $K_S \leq (\operatorname{card} K_{i_0 n/m})/2$. In this case, we work with the first term on the right-hand side of (5.8). Since $v_i|_{L_i} = v_{i-1}|_{L_i}$ for each positive integer i, we may define a continuous function $V:[L_0,L_{i_0}]\to\mathbb{R}$ such that $V=v_i$ on $[L_i,L_{i+1}]$ for $0\leq i\leq i_0-1$. Similarly, $P:[L_0,L_{i_0}]\to[0,\infty]$ defined by $P=\rho_i$ on $[L_i,L_{i+1}]$, $0\leq i\leq i_0-1$, is an upper gradient of V. Using V and P, we may interpret the first term on the right-hand side of (5.8) as a "weighted capacity" in the following manner.

Recall the definition of $\delta>0$ from (5.1). The definition of P and Hölder's inequality show that

$$\int_{[L_0,L_{i_0}]} P^{q-\delta}\,d\mathcal{H}^2 \leq \sum_{i=0}^{i_0-1}\left(\int_{[L_i,L_{i+1}]}\rho_i^q\,d\mathcal{H}^2\right)^{(q-\delta)/q}\left(\mathcal{H}^2([L_i,L_{i+1}])\right)^{\delta/q}. \tag{5.10}$$

We may cover $[L_i,L_{i+1}]$ by a rectangle of height 2 and width $w_{i+1}-w_{i+2}$, showing that

$$\mathcal{H}^2([L_i,L_{i+1}])\lesssim w_{i+1}-w_{i+2}\leq 3^{-in}.$$

From these estimates and application of Hölder's inequality to the sum, we see that

$$\begin{aligned}\int_{[L_0,L_{i_0}]} P^{q-\delta}\,d\mathcal{H}^2 &\lesssim \sum_{i=0}^{i_0-1}\left(2^{(1-q)i}\int_{[L_i,L_{i+1}]}\rho_i^q\,d\mathcal{H}^2\right)^{(q-\delta)/q} 2^{i(q-1)(q-\delta)/q}3^{-in\delta/q}\\ &\lesssim \left(\sum_{i=0}^{i_0-1}2^{(1-q)i}\int_{[L_i,L_{i+1}]}\rho_i^q\,d\mathcal{H}^2\right)^{(q-\delta)/q}\left(\sum_{i=0}^{i_0-1}2^{\left(\frac{(q-1)(q-\delta)}{\delta}-\frac{n}{\log_3 2}\right)i}\right)^{\delta/q}\end{aligned}$$

The number δ is defined so that

$$\left(\frac{(q-1)(q-\delta)}{\delta}-\frac{n}{\log_3 2}\right)=-n<0.$$

From this, we see that

$$\left(\int_{[L_0,L_{i_0}]} P^{q-\delta}\, d\mathcal{H}^2\right)^{q/(q-\delta)} \lesssim \sum_{i=0}^{i_0-1} 2^{(1-q)j} \int_{[L_i,L_{i+1}]} \rho_i^q \, d\mathcal{H}^2. \tag{5.11}$$

This estimate shows how the right-hand term above may be thought of as a capacity.

For each $k \in K_{i_0 n/m}$, set

$$E_k' = \{(x,y) \in W_k \cap L_{i_0} : V(x,y) \geq 1/2\}.$$

Since $V|_{L_{i_0}} = v_{i_0}$, we see that

$$E_k' \cup E_k = W_k \cap L_{i_0}.$$

As we have assumed that card $K_S \leq (\text{card}\, K_{i_0 n/m})/2$, it follows from Lemma 3.2 that

$$\mathcal{H}_\infty^{(\log_3 2)/m}\left(\bigcup_{k \notin K_S} E_k'\right) \gtrsim 1.$$

Moreover, since $i_0 \geq 2$, we see that

$$\mathcal{H}_\infty^{(\log_3 2)/m}(L_0 \cap X) \approx 1 \approx (\text{diam}[L_0, L_{i_0}])^{(\log_3 2)/m},$$

and $V|_{L_0} = v|_{L_0} \leq 0$. The inequalities in (5.6) show that $1 \leq (q-\delta) \leq 2$ and that $2 - (q - \delta) < (\log_3 2)/m \leq 2$. By Proposition 4.8, the space $[L_0, L_{i_0}]$ is Ahlfors 2-regular and supports a $(q-\delta)$-Poincaré inequality. Thus we may apply Theorem 2.10 to the space $[L_0, L_{i_0}]$, using the sets $L_0 \cap X$ and $\bigcup_{k \notin K_S} E_k'$, and the function V. We conclude that

$$\int_{[L_0,L_{i_0}]} P^{q-\delta} d\mathcal{H}^2 \gtrsim \text{diam}([L_0, L_{i_0}])^{2-(q-\delta)} \gtrsim 1.$$

Noting that the exponent $q/(q-\delta)$ depends only on n and m, inequalities (5.8) and (5.11) now yield the desired conclusion. □

Acknowledgments

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