

# SPACES OF SMALL METRIC COTYPE

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ABSTRACT. Naor and Mendel's metric cotype extends the notion of the Rademacher cotype of a Banach space to all metric spaces. Every Banach space has metric cotype at least 2. We show that any metric space that is bi-Lipschitz equivalent to an ultrametric space has infimal metric cotype 1. We discuss the invariance of metric cotype inequalities under snowflaking mappings and Gromov-Hausdorff limits, and use these facts to establish a partial converse of the main result.

## 1. INTRODUCTION

Recent connections between theoretical computer science, geometric functional analysis, and analysis on metric spaces have led to significant progress in these fields [4], [15], and [13]. Type and cotype inequalities, which play an important role in the geometry of Banach spaces (see e.g., [10], [7]), are one aspect of this synthesis. In [13], Mendel and Naor introduced a satisfactory notion of cotype for general metric spaces, and used it to solve a variety of problems regarding geometric embeddings. Among other important results, Naor and Mendel establish a non-linear version of the Maurey-Pisier theorem for spaces with no metric cotype. However, several results in [13] apply only to Banach spaces, though the statements are sensible in a general metric setting. The prospect of extending these results is tantalizing. In this paper, we begin this process by examining situations in which the theory of metric cotype differs from its linear counter-part.

Recall that a Banach space  $X$  is said to have Rademacher type  $p \in [1, 2]$  if there exists a number  $T \geq 1$  such that for each positive integer  $n$  and each  $x_1, x_2, \dots, x_n \in X$ ,

$$\sum_{i=1}^n \|x_i\|_X^p \geq T^{-p} \mathbb{E}_\epsilon \left\| \sum_{i=1}^n \epsilon_i x_i \right\|_X^p,$$

where the expectation is taken over uniformly distributed  $\epsilon \in \{-1, 1\}^n$ . Similarly,  $X$  is said to have Rademacher cotype  $q \in [2, \infty)$  if there exists a number  $C \geq 1$  such that for each positive integer  $n$  and each  $x_1, x_2, \dots, x_n \in X$ ,

$$\sum_{i=1}^n \|x_i\|_X^q \leq C^q \mathbb{E}_\epsilon \left\| \sum_{i=1}^n \epsilon_i x_i \right\|_X^q$$

where the expectation is as before.

We now describe the notion of metric cotype established in [13]. Let  $X$  be a metric space,  $n \in \mathbb{N}$ , and  $m \in 2\mathbb{N}$ . We denote the  $n$ -fold product of the the integers modulo  $m$  by  $\mathbb{Z}_m^n$ . For a function  $f: \mathbb{Z}_m^n \rightarrow X$ , an index  $1 \leq j \leq n$ , and a vector  $\delta \in \{-1, 0, 1\}^n$ , define

$$f_j(\epsilon) = f\left(\epsilon + \frac{m}{2}\mathbf{e}_j\right) \quad \text{and} \quad f_\delta(\epsilon) = f(\epsilon + \delta),$$

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where  $\mathbf{e}_j$  denotes the  $j$ th standard basis vector.

**Definition 1.1** (Mendel-Naor). *Let  $1 \leq p \leq q < \infty$ . A metric space  $(X, d)$  supports a  $(p, q)$  metric cotype inequality if there is constant  $\Gamma \geq 1$  and a scaling function  $m_{p,q}: \mathbb{Z} \rightarrow 2\mathbb{Z}$  with the following property. Given  $n \in \mathbb{Z}$  and setting  $m = m_{p,q}(n)$ , every mapping  $f: \mathbb{Z}_m^n \rightarrow X$  satisfies*

$$(1.1) \quad \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d(f(\epsilon), f_j(\epsilon))^p \leq \Gamma^p m^p n^{1-(p/q)} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1,0,1\}^n} d(f(\epsilon), f_\delta(\epsilon))^p.$$

*The expectations are taken over uniformly distributed  $\epsilon \in \mathbb{Z}_m^n$  and  $\delta \in \{-1,0,1\}^n$ . If  $(X, d)$  supports a  $(q, q)$  metric cotype inequality, then we say that it supports a metric cotype  $q$  inequality.*

Mendel and Naor's main result [13, Theorem 1.4] shows the aptness of the above definition.

**Theorem 1.2** (Mendel-Naor). *If a Banach space supports a  $(p, q)$  metric cotype inequality, then it has Rademacher cotype  $q'$  for every  $q' > q$ . Conversely, a Banach space with Rademacher cotype  $q$  supports a  $(p, q)$  metric cotype inequality for every  $1 \leq p \leq q$ .*

Motivated by these results, given a metric space  $(X, d)$ , we denote

$$q_X = \inf\{q \geq 1 : \text{there is } 1 \leq p \leq q \text{ such that } X \text{ supports a } (p, q) \text{ metric cotype inequality}\}.$$

In this notation, Theorem 1.2 implies that if  $X$  is a Banach space, then  $q_X$  is the infimum over  $q$  such that  $X$  has Rademacher cotype  $q$ .

Recall that if a Banach space has Rademacher cotype  $q$ , then it has Rademacher cotype  $q'$  for all  $q' \geq q$ , and that no Banach space supports a Rademacher cotype inequality with exponent less than 2. Our main result shows that spaces that certain highly disconnected metric spaces support "better" metric cotype inequalities than are possible for Banach spaces.

A metric space  $(X, d)$  is an *ultrametric space* if for all triples  $x, y, z \in X$ ,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Ultrametric spaces are fundamental mathematical objects, used in theoretical computer science [1],  $p$ -adic analysis [16], and mathematical biology [19]. Informally, they are spaces that arise as the leaves of a metric tree. Examples include all finite metric spaces, the standard middle third Cantor set, and the space  $\{2^{-i}\}_{i \in \mathbb{Z}} \subseteq \mathbb{R}$ . Every separable ultrametric space isometrically embeds into the Hilbert space of square summable sequences [17]. Hence, every such space supports a metric cotype 2 inequality. However, more is true.

**Theorem 1.3.** *Suppose that a metric space  $(X, d)$  is bi-Lipschitz equivalent to an ultrametric space. Then  $(X, d)$  supports a metric cotype  $q$  inequality for every  $q > 1$ . In particular,  $q_X = 1$ .*

The main tool in the proof of Theorem 1.3 is an isoperimetric inequality on a graph that corresponds to the right hand side of the metric cotype inequality (1.1). This graph can be thought of as an  $l^\infty$ -discrete torus. Similar isoperimetric inequalities have been shown on the  $l^1$ -discrete torus in [2], and our result may be of independent interest. The techniques in the proof of the isoperimetric inequality are also used to show that no metric space with at least two points can support a metric cotype 1 inequality.

The second portion of this paper discusses the invariance of metric cotype under various types of mappings between general metric spaces, and under limiting processes. We use the results regarding the invariance of cotype to work towards a converse of Theorem 1.3, employing the

formalism of  $s$ -snowflake spaces established by Tyson and Wu [18]. Let  $s \in [1, \infty]$ . A metric space  $(X, d)$  is said to be an  $L^s$ -metric space if for all triples  $x, y, z \in X$ ,

$$d(x, y) \leq \begin{cases} (d^s(x, z) + d^s(z, y))^{1/s} & 1 \leq s < \infty, \\ \max\{d(x, z), d(z, y)\} & s = \infty. \end{cases}$$

Note that a metric space is an  $L^\infty$ -metric space if and only if it is an ultrametric space. We define the *snowflake index*  $s_X$  of a metric space  $(X, d)$  by

$$s_X = \sup\{s : (X, d) \text{ is bi-Lipschitz equivalent to an } L^s\text{-metric space}\}.$$

A metric space  $(X, d)$  is said to be an  $s$ -snowflake if  $(X, d)$  is bi-Lipschitz equivalent to an  $L^s$ -metric space and  $s = s_X$ . Hence, a metric space is an  $\infty$ -snowflake if and only if it is bi-Lipschitz equivalent to an ultrametric. In this notation, Theorem 1.3 implies that if  $(X, d)$  is an  $\infty$ -snowflake, then  $q_X = 1$ .

Using results in [18] due to Laakso, we show the following statements, which are in the direction of a converse to Theorem 1.3.

**Theorem 1.4.** *Suppose that  $(X, d)$  is a 1-snowflake. Then  $q_X \geq 2$ .*

**Theorem 1.5.** *Suppose that  $(X, d)$  is a  $s$ -snowflake for some  $s < \infty$ . If  $1 \leq p \leq q < 2$ , then  $(X, d)$  does not support a  $(p, q)$  metric cotype inequality in which  $m_{p,q}(n) = O(n^{1/q})$ .*

As mentioned in [13], it is of interest to know if the conclusion of Theorem 1.5 implies that  $q_X \geq 2$ . Also, it would be interesting to know if the hypothesis in Theorem 1.5 can be weakened to the condition that  $s_X < \infty$ , and if the hypothesis in Theorem 1.3 can be weakened to the condition  $s_X = \infty$ .

Comparing Theorems 1.3 and 1.5 leads us to the following conjecture. Similar phenomena have been observed for conformal dimension; see [8].

**Conjecture 1.6.** *There is no metric space  $(X, d)$  for which  $1 < q_X < 2$ .*

It seems likely that a similar analysis could be made of the corresponding notion of metric type given in [12].

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## 2. CLASSES OF MAPPINGS

We now establish notation for the various classes of mappings we will consider throughout this paper. Let  $\phi: (Y, d_Y) \rightarrow (X, d_X)$  be a mapping between metric spaces.

For  $L \geq 1$ , we say that  $\phi$  is an  $L$ -bi-Lipschitz embedding if for all  $x, y \in X$ ,

$$\frac{d_Y(x, y)}{L} \leq d_X(\phi(x), \phi(y)) \leq Ld_Y(x, y).$$

Such a mapping distorts absolute distances by a fixed multiplicative factor.

For  $\alpha > 0$ , and  $L \geq 1$ , we say that  $\phi$  is an  $(\alpha, L)$ -snowflaking embedding if for all  $x, y \in Y$

$$\frac{d_Y(x, y)^\alpha}{L} \leq d_X(\phi(x), \phi(y)) \leq Ld_Y(x, y)^\alpha$$

When  $0 < \alpha \leq 1$ , such a mapping may be thought of as a bi-Lipschitz mapping defined on the snowflaked space  $(Y, d_Y^\alpha)$ . The term ‘‘snowflake’’ comes from the fact that the standard

parameterization of the Von Koch snowflake curve by the unit circle is a  $(\log_4 3, L)$ -snowflaking embedding for some  $L \geq 1$ .

Let  $\eta: [0, \infty) \rightarrow [0, \infty)$  be a homeomorphism. We say that  $\phi$  is an  $\eta$ -quasisymmetric embedding if for each triple of distinct points  $x, y, z \in Y$ ,

$$\frac{d_X(\phi(x), \phi(y))}{d_X(\phi(x), \phi(z))} \leq \eta \left( \frac{d_Y(x, y)}{d_Y(x, z)} \right).$$

Such a mapping distorts relative distances by a controlled amount. Note that every snowflaking embedding is quasisymmetric. Quasisymmetric embeddings are a generalization of conformal mappings to the metric space setting. For an introduction to their basic properties, see [6].

For  $c > 0$ , we say that the identity map  $\text{id}: (X, d_X) \rightarrow (X, cd_X)$  is a  $c$ -scaling. If  $\phi$  is an  $L$ -bi-Lipschitz embedding,  $(\alpha, L)$ -snowflaking embedding, or  $\eta$ -quasisymmetric embedding, then we say that the post-composition of  $\phi$  with a  $c$ -scaling is a *scaled*  $L$ -bi-Lipschitz embedding,  $(\alpha, L)$ -snowflaking embedding, or  $\eta$ -quasisymmetric embedding, respectively. The reason for this notation is that the value  $c$  is often irrelevant, while the parameters associated to the other classes of mappings are not.

### 3. $\infty$ -SNOWFLAKES

We first provide a characterization of  $\infty$ -snowflakes similar to that described in [18, Section 2] and [6, Section 14.24].

**Proposition 3.1.** *Let  $(X, d)$  be a metric space. Then the following conditions are equivalent:*

- (i) *the space  $(X, d)$  is  $L$ -bi-Lipschitz equivalent to an ultrametric space,*
- (ii) *the space  $(X, d)$  is  $\eta$ -quasisymmetrically equivalent to an ultrametric space,*
- (iii) *there is a constant  $C \geq 1$  such that for every subset  $S$  of  $X$  satisfying  $0 < \text{diam } S < \infty$ , there is a subset  $A \subseteq S$  such that*

$$\text{dist}(A, S \setminus A) \geq \frac{\text{diam } S}{C}.$$

- (iv) *there is a constant  $C \geq 1$  such that for every subset  $S$  of  $X$  satisfying  $1 < \text{card } S < \infty$ , there is a subset  $A \subseteq S$  such that*

$$\text{dist}(A, S \setminus A) \geq \frac{\text{diam } S}{C}.$$

*Remark 3.2.* The above proposition is quantitative in the following sense. If condition (i) is satisfied, then condition (ii) is satisfied with  $\eta(t) = L^2 t$ . If condition (ii) is satisfied, then conditions (iii) and (iv) are both satisfied with  $C = 2\eta(1)$ . If condition (iii) or (iv) is satisfied, then condition (i) is satisfied with  $L = C$ .

A space satisfying condition (iii) above is said to be  $C$ -uniformly disconnected. For ease of notation, let us call condition (iv) above the  $C$ -finite separation property. We prove Proposition 3.1 via three easy lemmas. We leave the proof of the first to the reader, and the proof of the third is essentially the same as the proof of [5, Proposition 15.7], so we omit it as well.

**Lemma 3.3.** *If  $f: X \rightarrow Y$  is an  $\eta$ -quasisymmetric homeomorphism, and  $X$  is  $C$ -uniformly disconnected for some  $C \geq 1$ , then  $Y$  is  $2\eta(C)$ -uniformly disconnected.*

**Lemma 3.4.** *If  $(X, d)$  is an ultrametric space, then it is 1-uniformly disconnected.*

*Proof.* Let  $S$  be a subset of  $X$  satisfying  $0 < \text{diam } S < \infty$ . Set  $D = \text{diam } S$ , and let  $\mathcal{D}$  be a maximal collection of points in  $S$  such that  $d(x, y) = D$  for all  $x, y \in \mathcal{D}$ . The definition of diameter implies that  $\mathcal{D}$  has at least two points. Fix any  $x_0 \in \mathcal{D}$ , and define

$$A = \{x \in S : d(x_0, x) < \text{diam } S\}.$$

Let  $x \in A$  and  $y \in S \setminus A$ . Since  $\mathcal{D}$  is maximal, there exists  $y_0 \in \mathcal{D}$  such that  $d(y_0, y) < D$ . Noting that

$$D = d(x_0, y_0) \leq \max\{d(x_0, x), d(x, y_0)\},$$

we see that  $d(x, y_0) = D$ . Similarly,

$$D = d(x, y_0) \leq \max\{d(x, y), d(y, y_0)\},$$

and so  $d(x, y) = D$ . This implies the result.  $\square$

**Lemma 3.5.** *Suppose that a metric space  $(X, d)$  satisfies the  $C$ -finite separation property. Then there is an ultrametric space that is  $C$ -bi-Lipschitz equivalent to  $X$ .*

*Proof of Proposition 3.1.* An  $L$ -bi-Lipschitz mapping is  $\eta$ -quasisymmetric with  $\eta(t) = L^2 t$ . Thus (i) implies (ii). Lemmas 3.3 and 3.4 show that (ii) implies (iii). Clearly (iii) implies (iv). Finally, Lemma 3.5 shows that (iv) implies (i).  $\square$

The conditions in Proposition 3.1 provide a natural tree structure on any bounded  $\infty$ -snowflake. Let  $\mathcal{S}$  denote the collection of finite sequences  $\alpha = \{\alpha_i\}_{i=1}^k$  such that  $\alpha_i \in \{0, 1\}$  for each index  $i$ . We also consider the empty sequence  $\emptyset$  to be in  $\mathcal{S}$ . Given  $\alpha \in \mathcal{S}$  and  $\delta \in \{0, 1\}$ , we set

$$\alpha, \epsilon = (\alpha_1, \dots, \alpha_k, \delta) \in \mathcal{S}.$$

**Definition 3.6.** *A  $C$ -separated tree structure,  $C \geq 1$ , on a metric space  $(X, d)$  is a collection  $\{A_\alpha\}_{\alpha \in \mathcal{S}}$  of subsets of  $X$  such that*

- (1)  $A_\emptyset = X$ ,
- (2)  $A_\alpha = A_{\alpha,0} \cup A_{\alpha,1}$  for each  $\alpha \in \mathcal{S}$ ,
- (3)  $\text{diam } A_\alpha \leq C \text{dist}(A_{\alpha,0}, A_{\alpha,1})$  for all  $\alpha \in \mathcal{S}$ .

Here we employ the convention that the distance from any set to the empty set is 0. Hence, in a separated tree structure, the decomposition  $A_\alpha = A_{\alpha,0} \cup A_{\alpha,1}$  is non-trivial unless  $A_\alpha$  is a single point or the empty set.

Clearly, a metric space supports a  $C$ -separated tree structure if and only if it is bounded and  $C$ -uniformly disconnected.

We will use the following basic result regarding separated tree structures. We say that  $\{\alpha_i\}_{i=1}^k \in \mathcal{S}$  is an *ancestor* of  $\{\alpha'_i\}_{i=1}^{k'} \in \mathcal{S}$  if  $k \leq k'$  and  $\alpha_j = \alpha'_j$  for  $j \leq k$ .

**Lemma 3.7.** *Let  $\{A_\alpha\}_{\alpha \in \mathcal{S}}$  be a  $C$ -separated tree structure. If  $\alpha$  and  $\alpha'$  are elements of  $\mathcal{S}$  such that neither is an ancestor of the other, then*

$$\text{dist}(A_\alpha, A_{\alpha'}) \geq \frac{\max\{\text{diam } A_\alpha, \text{diam } A_{\alpha'}\}}{C}.$$

*Proof.* Let  $\beta \in \mathcal{S}$  be the longest sequence that is an ancestor of both  $\alpha$  and  $\alpha'$ . The result is immediate if  $\alpha = \alpha'$ . Otherwise, we may assume without loss of generality that  $\beta, 0$  is an ancestor of  $\alpha$  and  $\beta, 1$  is an ancestor of  $\alpha'$ . Then

$$A_\alpha \subseteq A_{\beta,0} \subseteq A_\beta \quad \text{and} \quad A_{\alpha'} \subseteq A_{\beta,1} \subseteq A_\beta.$$

From this and the definition of a  $C$ -separated tree structure, we see that

$$\text{dist}(A_\alpha, A_{\alpha'}) \geq \text{dist}(A_{\beta,0}, A_{\beta,1}) \geq \frac{\text{diam}(A_\beta)}{C} \geq \frac{\max\{\text{diam } A_\alpha, \text{diam } A_{\alpha'}\}}{C},$$

as desired.  $\square$

#### 4. THE PROOF OF THEOREM 1.3

For a positive integer  $n$  and an even positive integer  $m$ , we denote by  $L_m^n$  the graph with vertex set  $\mathbb{Z}_m^n$  and edge set

$$E_L = \{(\epsilon, \epsilon') : \epsilon' = \epsilon + \frac{m}{2}\mathbf{e}_j \text{ for some } j = 1, 2, \dots, n\}.$$

Similarly, we denote by  $R_m^n$  the graph with vertex set  $\mathbb{Z}_m^n$  and edge set

$$E_R = \{(\epsilon, \epsilon') : \epsilon' = \epsilon + \delta \text{ for some } 0 \neq \delta \in \{-1, 0, 1\}^n\}.$$

As the graph  $R_m^n$  is connected, it defines a path metric  $d_{\mathbb{Z}_m^n}$  on  $\mathbb{Z}_m^n$ . That is, the distance between two points of  $\mathbb{Z}_m^n$  is the length of the shortest path connecting them in  $R_m^n$  when each edge is assumed to have length 1. Given a set  $A \subseteq \mathbb{Z}_m^n$ , the *outer boundary* of  $A$  is defined by

$$(4.1) \quad \partial A = \{\epsilon \in \mathbb{Z}_m^n : d_{\mathbb{Z}_m^n}(A, \epsilon) = 1\}.$$

The following isoperimetric inequality for  $R_m^n$  allows us to give a lower bound for the right-hand side of (1.1). Note that it is of Euclidean type. We defer the proof to the end of Section 5.

**Theorem 4.1.** *Suppose that  $A \subset \mathbb{Z}_m^n$  satisfies  $|A| \leq (m-1)^n$ . Then*

$$|\partial A| \geq 2|A|^{(n-1)/n}.$$

*Proof of Theorem 1.3.* We assume that there is  $L \geq 1$  such that  $X$  is  $L$ -bi-Lipschitz equivalent to an ultrametric space. Let  $q > 1$ , and let  $m_{q,q} : \mathbb{Z} \rightarrow 2\mathbb{Z}$  be any function such that

$$m_{q,q}(n) \geq \max\{(n3^n)^{1/(q-1)}, (1 - 2^{-1/n})^{-1}\}.$$

Fix  $n \in \mathbb{Z}$  and set  $m = m_{q,q}(n)$ . We will show that for any  $f : \mathbb{Z}_m^n \rightarrow X$ ,

$$(4.2) \quad \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d(f(\epsilon), f_j(\epsilon))^q \leq \Gamma^q m^q \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} d(f(\epsilon), f_\delta(\epsilon))^q,$$

for some  $\Gamma \geq 1$  that depends only on  $L$ .

As a subset of  $X$ , the set  $f(\mathbb{Z}_m^n)$  is also  $L$ -bi-Lipschitz equivalent to an ultrametric space. Proposition 3.1 and Remark 3.2 now imply that there is a  $C$ -separated tree structure  $\{A_\alpha\}_{\alpha \in \mathcal{S}}$  on  $f(\mathbb{Z}_m^n)$ , where  $C = 2L^2$ .

Since  $f(\mathbb{Z}_m^n)$  is a finite set, we may find  $i_0 \in \mathbb{N}$  such that if the length of  $\alpha$  is greater than  $i_0$ , then  $A_\alpha$  is either empty or a singleton. For each  $\alpha \in \mathcal{S}$ , set  $F_\alpha = f^{-1}(A_\alpha)$ , with the convention that  $F_\alpha = \emptyset$  if  $A_\alpha = \emptyset$ .

The graph  $L_m^n$  is related to the left-hand side of the metric cotype inequality (4.2) in the following way:

$$(4.3) \quad \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d(f(\epsilon), f_j(\epsilon))^q = 2m^{-n} \sum_{(\epsilon, \epsilon') \in E_L} d_X(f(\epsilon), f(\epsilon'))^q.$$

The factor of 2 occurs because the graph  $L_m^n$  is not directed.

We estimate the quantity in (4.3) by partitioning  $E_L$  into sets where uniform estimates are possible. For an integer  $i \geq 0$ , we denote by  $\mathbf{1}_i \in \mathcal{S}$  the string consisting of  $i$  ones, and set

$$E_L^i = \{(\epsilon, \epsilon') \in E_L : \{\epsilon, \epsilon'\} \subseteq F_{\mathbf{1}_i} \text{ and } \{\epsilon, \epsilon'\} \cap F_{\mathbf{1}_{i,0}} \neq \emptyset\}.$$

Since there are at most  $n$  edges attached to any given vertex in  $L_m^n$ , we estimate that  $|E_L^i| \leq n|F_{\mathbf{1}_{i,0}}|$ , and so

$$\sum_{(\epsilon, \epsilon') \in E_L^i} d_X(f(\epsilon), f(\epsilon'))^q \leq n|F_{\mathbf{1}_{i,0}}|(\text{diam } A_{\mathbf{1}_i})^q.$$

Moreover,  $\{E_L^i\}_{i=0}^{i_0}$  exhausts  $E_L$ , and  $\text{diam } A_{\mathbf{1}_{i_0}} = 0$ . Thus, (4.3) implies that

$$(4.4) \quad \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d(f(\epsilon), f_j(\epsilon))^q \leq m^{-n} \sum_{i=0}^{i_0-1} 2n|F_{\mathbf{1}_{i,0}}|(\text{diam } A_{\mathbf{1}_i})^q.$$

We now give a lower bound for the right-hand side of (4.2). We relate it to the graph  $R_m^n$  via the identity

$$(4.5) \quad \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1,0,1\}^n} d(f(\epsilon), f_\delta(\epsilon))^q = 2(3m)^{-n} \sum_{(\epsilon, \epsilon') \in E_R} d_X(f(\epsilon), f(\epsilon'))^q.$$

For an integer  $i \geq 0$ , define

$$E_R^i = \{(\epsilon, \epsilon') \in E_R : |\{\epsilon, \epsilon'\} \cap F_{\mathbf{1}_{i,0}}| = 1\}.$$

An edge in  $E_R$  appears in at most two of the sets  $E_R^0, \dots, E_R^{i_0-1}$ . Hence

$$(4.6) \quad \sum_{(\epsilon, \epsilon') \in E_R} d_X(f(\epsilon), f(\epsilon'))^q \geq \frac{1}{2} \sum_{i=0}^{i_0-1} \sum_{(\epsilon, \epsilon') \in E_R^i} d_X(f(\epsilon), f(\epsilon'))^q.$$

Let  $(\epsilon, \epsilon') \in E_R^i$  for some integer  $i \geq 0$ . We claim that

$$(4.7) \quad d_X(f(\epsilon), f(\epsilon')) \geq \frac{\text{diam } A_{\mathbf{1}_i}}{C}.$$

Assume without loss of generality that  $\epsilon \in F_{\mathbf{1}_{i,0}}$  and  $\epsilon' \notin F_{\mathbf{1}_{i,0}}$ . If  $\epsilon' \in F_{\mathbf{1}_{i+1}}$ , then (4.7) follows from the definition of a  $C$ -separated tree structure. If  $\epsilon' \notin F_{\mathbf{1}_{i+1}}$ , then  $\epsilon' \notin F_{\mathbf{1}_i}$ . This implies that we may find  $\alpha \in \mathcal{S}$  such that  $\epsilon' \in F_\alpha$  and neither  $\alpha$  nor  $\mathbf{1}_i$  is an ancestor of the other. Since  $\epsilon \in F_{\mathbf{1}_i}$ , Lemma 3.7 now yields (4.7).

Since  $|E_R^i| \geq |\partial F_{\mathbf{1}_{i,0}}|$  for each integer  $i \geq 0$ , the inequalities (4.5), (4.6), and (4.7) show that

$$(4.8) \quad C^q m^q \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1,0,1\}^n} d(f(\epsilon), f_\delta(\epsilon))^q \geq m^{-n} \sum_{i=0}^{i_0-1} 3^{-n} m^q |\partial F_{\mathbf{1}_{i,0}}| (\text{diam } A_{\mathbf{1}_i})^q.$$

The desired inequality (4.2) with  $\Gamma = C$  will now follow from (4.4) and (4.8) provided we additionally show that

$$(4.9) \quad 3^{-n} m^q |\partial F_{\mathbf{1}_{i,0}}| \geq 2n|F_{\mathbf{1}_{i,0}}|.$$

For each integer  $0 \leq i < i_0$ , the sets  $F_{\mathbf{1}_{i,0}}$  and  $F_{\mathbf{1}_{i+1}}$  are disjoint subsets of  $\mathbb{Z}_m^n$ . Thus, by relabeling the separated tree structure if needed, we may assume without loss of generality that for each integer  $0 \leq i < i_0$ , the set  $F_{\mathbf{1}_{i,0}}$  has cardinality no greater than  $m^n/2$ . It follows from the definition of  $m_{q,q}$  that  $(m-1)^n \geq m^n/2$ . Hence by Theorem 4.1, for each integer  $0 \leq i < i_0$ ,

$$|\partial F_{\mathbf{1}_{i,0}}| \geq 2|F_{\mathbf{1}_{i,0}}|^{(n-1)/n}.$$

The definition of  $m_{q,q}$  also implies that  $3^{-n}m^q \geq mn$ . Hence, using the above isoperimetric inequality and the trivial estimate  $|F_{\mathbf{1}_i,0}| \leq m^n$ , we see that

$$3^{-n}m^q|\partial F_{\mathbf{1}_i,0}| \geq (2n|F_{\mathbf{1}_i,0}|) \left( m|F_{\mathbf{1}_i,0}|^{-1/n} \right) \geq 2n|F_{\mathbf{1}_i,0}|,$$

as desired.  $\square$

We record the following corollary, which one might consider to be a very weak non-linear version of the Maurey-Pisier theorem for cotype 1. In light of Conjecture 1.6, we suspect that a stronger statement is true.

**Definition 4.2.** *Let  $\epsilon > 0$ , and let  $a, b \in X$ . An  $\epsilon$ -chain in  $X$  connecting  $a$  to  $b$  is a finite sequence of points  $x_0 = a, x_1, \dots, x_k = b$  such that for each  $i = 0, \dots, k-1$ ,*

$$d(x_i, x_{i+1}) < \epsilon.$$

**Corollary 4.3.** *Suppose that  $(X, d)$  is a metric space with  $q_X > 1$ . Then for all  $C \geq 1$ , there are points  $a, b \in X$  and a  $d(a, b)/C$ -chain connecting  $a$  to  $b$  with diameter  $d(a, b)$ .*

*Proof.* Fix  $C \geq 1$ . By Theorem 1.3 and Proposition 3.1, the family  $\mathcal{F}$  of finite subsets  $T \subseteq X$  such that if  $A \neq T$  is a non-empty subset of  $T$  then

$$(4.10) \quad \text{dist}(A, T \setminus A) < \frac{\text{diam } T}{C}$$

is nonempty.

Let  $S \in \mathcal{F}$  and set  $M := \text{diam } S$ . Since  $S$  is finite, we may find points  $a, b \in S$  such that  $d(a, b) = M$ . Consider the set

$$\mathcal{C} := \{x \in S : \text{there exists an } (M/C)\text{-chain in } S \text{ connecting } a \text{ to } x\}.$$

Clearly  $\mathcal{C}$  is non-empty as it contains  $a$ . If  $\mathcal{C} \neq S$ , then (4.10) provides points  $x \in \mathcal{C}$  and  $y \notin \mathcal{C}$  such that  $d(x, y) < M/C$ . Adding the point  $y$  to the  $(M/C)$ -chain connecting  $a$  to  $x$  yields an  $(M/C)$ -chain connecting  $a$  to  $y$ , a contradiction. Thus  $\mathcal{C} = S$ , and so there is an  $(M/C)$ -chain  $\{x_0, \dots, x_\alpha\}$  in  $S$  connecting  $a$  to  $b$ .  $\square$

## 5. AN ISOPERIMETRIC INEQUALITY FOR $R_m^n$

In this section we prove Theorem 4.1. Many of the ideas in this section are extensions of ideas in [9] and [2]. Throughout this section, we let  $m$  be a fixed positive even integer. Moreover, in this section only we typically denote elements of the group  $\mathbb{Z}_m^n$  by Roman characters such as  $x$  and  $y$  rather than by Greek characters such as  $\epsilon$ , as was done in the previous section.

Our main tool will be an order  $<_n$  on the  $\mathbb{Z}_m^n$ , the vertices of  $R_m^n$ . For  $n = 1$ , we order the set  $\{0, 1, \dots, m-1\}$  by first declaring that

$$0 <_1 1 <_1 2 <_1 \dots <_1 m/2,$$

and then declaring that for  $k = 1, \dots, m/2 - 1$ ,

$$k <_1 m - k <_1 k + 1.$$

For example, when  $m = 10$ ,

$$0 <_1 1 <_1 9 <_1 2 <_1 8 <_1 3 <_1 7 <_1 4 <_1 6 <_1 5.$$

We denote the immediate successor of a number  $k \in \mathbb{Z}_m \setminus \{m/2\}$  in the order  $<_1$  by  $s_1(k)$ .

For a general  $n \geq 1$  and  $x = (x_1, \dots, x_n) \in \mathbb{Z}_m^n$ , we denote

$$|x|_\infty = \max_{i=1, \dots, n} x_i,$$



where the maximum is taken with respect to the ordering  $<_1$ , and

$$i_x = \min\{i : x_i = |x|_\infty\},$$

where the minimum is taken with respect to the standard ordering on  $\{1, \dots, n\}$ . That is,  $i_x$  is the left-most index where the maximum value with respect to  $<_1$  is achieved.

For  $n \geq 2$ , denote

$$\hat{x} = (x_1, \dots, x_{i_x-1}, x_{i_x+1}, \dots, x_n) \in \mathbb{Z}_m^{n-1}.$$

Suppose that the ordering  $<_{n-1}$  is defined, and let  $x, y$  in  $\mathbb{Z}_m^n$ . If  $|x|_\infty <_1 |y|_\infty$ , then we declare that  $x <_n y$ . If  $|x|_\infty = |y|_\infty$  and  $i_x > i_y$ , then we declare  $x <_n y$  as well. Finally, if  $|x|_\infty = |y|_\infty$  and  $i_x = i_y$ , then we declare that  $x <_n y$  if and only if  $\hat{x} <_{n-1} \hat{y}$ . With these specifications,  $<_n$  is a total order on  $\mathbb{Z}_m^n$ .

The main result of this section is the following isoperimetric inequality, which provides more detailed information than Theorem 4.1. Recall that the outer boundary  $\partial A$  of a subset  $A \subseteq \mathbb{Z}_m^n$  in the graph  $R_n^m$  was defined by (4.1).

**Theorem 5.1.** *Let  $A \subseteq \mathbb{Z}_m^n$ . Then  $|\partial A| \geq |\partial I|$ , where  $I$  is the initial segment of length  $|A|$  in  $(\mathbb{Z}_m^n, <_n)$ .*

We begin by describing immediate successors in  $(\mathbb{Z}_m^n, <_n)$ . It easily follows from the definitions that  $(m/2, \dots, m/2)$  is the maximum element of  $(\mathbb{Z}_m^n, <_n)$ . For any  $x \in \mathbb{Z}_m^n$ , we set

$$|x|_0 = \min_{i=1, \dots, n} x_i,$$

where the minimum is taken with respect to the ordering  $<_1$ , and define

$$j_x = \max\{j : x_j = |x|_0\},$$

where the maximum is taken with respect to the standard ordering on  $\{1, \dots, n\}$ . That is,  $j_x$  is the right-most index where the minimum value with respect to  $<_1$  is achieved. Familiarity with the ordering  $<_n$  results in the following proposition.

**Proposition 5.2.** *Let  $x \in \mathbb{Z}_m^n \setminus \{(m/2, \dots, m/2)\}$ , and define  $s_n(x) \in \mathbb{Z}_m^n$  by*

$$(5.1) \quad (s_n(x))_j = \begin{cases} 0 & x_j = |x|_0 \text{ and } j < j_x, \\ s_1(|x|_0) & j = j_x, \\ 0 & x_j = s_1(|x|_0) \text{ and } j > j_x, \\ x_j & \text{otherwise.} \end{cases}$$

*Then  $s_n(x)$  is the immediate successor of  $x$  in  $<_n$ .*

We will need the following related technical result. Given a vector  $x \in \mathbb{Z}_m^n$ ,  $n \geq 2$ , and an index  $i \in \{1, \dots, n\}$ , we denote by  $\pi_i(x) \in \mathbb{Z}_m^{n-1}$  the vector resulting from removing the  $i$ -th coordinate of  $x$ . In particular,  $\hat{x} = \pi_{i_x}(x)$ .

**Lemma 5.3.** *Let  $n \geq 2$ . Fix an index  $i \in \{1, \dots, n\}$ , and suppose that  $x, y \in \mathbb{Z}_m^n$  satisfy  $x_i = y_i$ . Then  $x <_n y$  if and only if  $\pi_i(x) <_{n-1} \pi_i(y)$ .*

The proof of Lemma 5.3 is left to the reader.

To prove Theorem 5.1, it is more convenient to work with neighborhoods than boundaries. For a set  $A \subset \mathbb{Z}_m^n$ , we denote

$$\mathcal{N}A = \{x \in \mathbb{Z}_m^n : \text{dist}_{\mathbb{Z}_m^n}(A, x) \leq 1\} = A \cup \partial A.$$

The following notation will be used to relate the order  $<_n$  to the metric  $d_{\mathbb{Z}_m^n}$ . Let  $x \in \mathbb{Z}_m^n$ . For each  $i = 1, \dots, n$ , denote

$$x_i^+ = \max\{x_i + 1, x_i, x_i - 1\} \quad \text{and} \quad x_i^- = \min\{x_i + 1, x_i, x_i - 1\},$$

where the maximum and minimum are taken with respect to the ordering  $<_1$ . Note that  $x_i <_1 x_i^+$  unless  $x_i = m/2$ , and  $x_i^- <_1 x_i$  unless  $x_i = 0$ . Define  $x^+ \in \mathbb{Z}_m^n$  by

$$x^+ = (x_1^+, \dots, x_n^+),$$

and similarly define  $x^-$ . Finally, denote  $x_i^\pm = \{x_i, x_i^+, x_i^-\}$ . With this notation,

$$\mathcal{N}(\{x\}) = \{y \in \mathbb{Z}_m^n : y_i \in x_i^\pm, i = 1, \dots, n\}.$$

**Lemma 5.4.** *Let  $S = \{x \in \mathbb{Z}_m^n : x \leq_n v\}$  be an initial segment in  $(\mathbb{Z}_m^n, <_n)$ . Then*

$$\mathcal{N}S = \{x \in \mathbb{Z}_m^n : x \leq_n v^+\}.$$

*Proof.* We induct on  $|S|$ . If  $|S| = 1$ , then  $S = \{(0, 0, \dots, 0)\}$  and

$$\mathcal{N}S = \{0, 1, m-1\}^n = \{x \in \mathbb{Z}_m^n : x \leq_n (0, \dots, 0)^+\}.$$

Now, let  $v \in \mathbb{Z}_m^n \setminus \{(m/2, \dots, m/2)\}$ , and let  $S = \{x \in \mathbb{Z}_m^n : x \leq_n v\}$  be the corresponding initial segment. We assume  $\mathcal{N}S$  is the initial segment  $\{x \in \mathbb{Z}_m^n : x \leq_n v^+\}$ , and we will show that

$$(5.2) \quad \mathcal{N}(S \cup \{s_n(v)\}) = \{x \in \mathbb{Z}_m^n : x \leq_n s_n(v)^+\}.$$

Note that Proposition 5.2 implies that  $v^+ \leq_n s_n(v)^+$ . Hence, it follows from the induction hypothesis and the definitions that  $s_n(v)^+$  is the maximal vector in  $\mathcal{N}(S \cup \{s_n(v)\})$ , and so it suffices to show that  $\mathcal{N}(S \cup \{s_n(v)\})$  is an initial segment. If  $v^+$  is the maximal vector in  $(\mathbb{Z}_m^n, <_n)$ , then  $s_n(v)^+$  is maximal as well, and there is nothing to show. Thus we may assume that  $v^+$  is non-maximal.

Consider the set

$$\mathcal{N}\{s_n(v)\} = \{x \in \mathbb{Z}_m^n : x_i \in s_n(v)_i^\pm, i = 1, \dots, n\},$$

and let  $I = \{i \in \{1, \dots, n\} : s_n(v)_i = 0\}$ . For any subset  $J \subseteq \{1, \dots, n\} \setminus I$ , define  $y(J) \in \mathbb{Z}_m^n$  by

$$(y(J))_i = \begin{cases} s_n(v)_i & i \notin J, \\ s_n(v)_i^- & i \in J. \end{cases}$$

Then  $y(J) \leq_n s_n(v)$ . As  $v$  is not maximal,  $v <_n v^+$ . Since  $s_n(v)$  is the immediate successor of  $v$ , these inequalities imply that  $y(J) \leq_n v^+$ , and hence  $y(J) \in \mathcal{N}S$  by the inductive assumption. Noting that for  $i \notin I$ ,

$$s_n(v)_i^\pm \setminus (s_n(v)_i^-)^\pm = \{s_n(v)_i^+\},$$

we now see that

$$(5.3) \quad \mathcal{N}\{s_n(v)\} \setminus \mathcal{N}S \subseteq \{x \in \mathbb{Z}_m^n : x_i = s_n(v)_i^+ \text{ for } i \notin I, \text{ and } x_i \in \{0, 1, m-1\} \text{ for } i \in I\}.$$

Let  $A$  denote the set on the right hand side of equation (5.3). We claim that  $A$  consists precisely of the  $3^{|I|}$  successors of  $v^+$  (recall that we have assumed that  $v^+$  is non-maximal). Since  $A \subseteq \mathcal{N}\{s_n(v)\}$ , the claim and the inductive assumption imply that  $\mathcal{N}(S \cup \{s_n(v)\})$  is an initial segment, as desired.

To prove the claim, we will show that for  $i = 1, \dots, n$ ,

$$(5.4) \quad s_n(v^+)_i = \begin{cases} s_n(v)_i^+ & i \notin I, \\ 0 & i \in I. \end{cases}$$

The claim then follows easily Proposition 5.2.

As  $v^+$  is non-maximal, we see that  $|v^+|_0 = |v|_0^+ \neq m/2$ , and that  $j_{v^+} = j_v$ . Moreover, given  $k, l \in \mathbb{Z}_m \setminus \{m/2\}$ , if  $k^+ = l^+$  then  $k = l$ . Similarly, if  $k^+$  is not maximal, then  $s_1(k)^+ = s_1(k^+)$ . Applying these facts to Proposition 5.2, we now see that

$$\begin{aligned} s_n(v^+)_i &= \begin{cases} 0 & v_i^+ = |v^+|_0 \text{ and } i < j_{v^+}, \\ s_1(|v^+|_0) & i = j_{v^+}, \\ 0 & v_i^+ = s_1(|v^+|_0) \text{ and } i > j_{v^+}, \\ v_i^+ & \text{otherwise,} \end{cases} \\ &= \begin{cases} 0 & v_i = |v|_0 \text{ and } i < j_v, \\ s_1(|v|_0)^+ & i = j_v, \\ 0 & v_i = s_1(|v|_0) \text{ and } i > j_v, \\ v_i^+ & \text{otherwise.} \end{cases} \end{aligned}$$

The representation (5.4) now follows from the formula for  $s_n(v)$  provided by Proposition 5.2.  $\square$

In the proof of Lemma 5.4, we have also shown the following statement.

**Lemma 5.5.** *Let  $S = \{x \in \mathbb{Z}_m^n : x \leq_n v\}$  be an initial segment that is not all of  $\mathbb{Z}_m^n$ , and let  $l$  be the number of coordinates of  $s_n(v)$  with value 0. Then*

$$|\mathcal{N}(S \cup \{s_n(v)\}) \setminus \mathcal{N}S| = 3^l.$$

Equivalently,

$$|\{x \in \mathbb{Z}_m^n : v^+ <_n x \leq_n s_n(v)^+\}| = 3^l.$$

In the proof of Theorem 5.1, we will employ the following notation. Given any subset  $S$  of  $\mathbb{Z}_m^n$ , a value  $k \in \mathbb{Z}_m$ , and an index  $i \in \{1, \dots, n\}$ , we define  $S_i^k$  to be the set of vectors in  $\mathbb{Z}_m^{n-1}$  such that the result of inserting the value  $k$  between the  $(i-1)$ -st and  $i$ -th coordinates is a vector in  $S$ . That is,

$$S_i^k = \{x \in \mathbb{Z}_m^{n-1} : (x_1, x_2, \dots, x_{i-1}, k, x_i, \dots, x_{n-1}) \in S\}$$

As each  $S_i^k$  is in bijection with the collection of vectors in  $S$  with  $i$ -th coordinate equal to  $k$ , we see that

$$(5.5) \quad \sum_{k \in \mathbb{Z}_m} |S_i^k| = |S|.$$

Moreover, it follows from the definitions that

$$(5.6) \quad (\mathcal{N}S)_i^k = \mathcal{N}(S_i^k) \cup \mathcal{N}(S_i^{k-1}) \cup \mathcal{N}(S_i^{k+1}).$$

*Proof of Theorem 5.1.* Let  $A \subseteq \mathbb{Z}_m^n$ . We wish to show that  $|\partial A|$  is at least as large as  $|\partial I_{|A|}|$ , where  $I_{|A|}$  is an initial segment of  $(\mathbb{Z}_m^n, <_n)$  of cardinality  $|A|$ .

If  $n = 1$ , then  $R_m^1$  is a topological circle. This easily implies that for any  $A \subseteq \mathbb{Z}_m$ ,

$$(5.7) \quad |\partial A| \geq \min\{2, |\mathbb{Z}_m \setminus A|\}.$$

The result now follows from noting that equality holds in (5.7) when  $A$  is an initial segment.

We proceed by induction. Suppose that  $n \geq 2$  and that the statement is valid for all dimensions less than  $n$ . The idea of the proof is to repeatedly alter  $A$  in a way that does not change its cardinality nor increase the cardinality of its neighborhood, and so that the alterations eventually reach an initial segment.

We denote the placement of a vector  $x$  in the ordered set  $(\mathbb{Z}_m^n, <_n)$  by  $p_n(x)$ , i.e.,

$$p_n((0, \dots, 0)) = 1 \quad \text{and} \quad p_n(s_n(x)) = p_n(x) + 1.$$

By definition, the initial segment  $I_{|A|}$  is the unique minimizer of the quantity

$$\sum_{x \in S} p_n(x)$$

among all subsets  $S \subseteq \mathbb{Z}_m^n$  with  $|S| = |A|$ . Thus, it suffices to show that if  $A$  is not an initial segment, then there exists a subset  $A' \subseteq \mathbb{Z}_m^n$  of the same cardinality as  $A$  that satisfies  $|\partial A'| \leq |\partial A|$  and

$$\sum_{x \in A'} p_n(x) < \sum_{x \in A} p_n(x).$$

Towards this goal, suppose that  $A$  is not an initial segment, and let  $i \in \{1, \dots, n\}$ . Define the *i-compression* of  $A$  to be the unique subset  $C_i$  of  $\mathbb{Z}_m^n$  with the property that for each  $k \in \mathbb{Z}_m$ , the set  $(C_i)_i^k$  is an initial segment in  $(\mathbb{Z}_m^{n-1}, <_{n-1})$  of length  $|A_i^k|$ . We say that  $A$  is *i-compressed* if it is equal to its *i-compression*.

Applying (5.5) yields that

$$|A| = \sum_{k \in \mathbb{Z}_m} |A_i^k| = \sum_{k \in \mathbb{Z}_m} |(C_i)_i^k| = |C_i|.$$

We claim that  $|\mathcal{N}(C_i)| \leq |\mathcal{N}A|$ . We see from (5.5) that

$$\sum_{k \in \mathbb{Z}_m} |(\mathcal{N}A)_i^k| = |\mathcal{N}A| \quad \text{and} \quad \sum_{k \in \mathbb{Z}_m} |(\mathcal{N}(C_i))_i^k| = |\mathcal{N}(C_i)|.$$

Thus it suffices to show that for each  $k \in \mathbb{Z}_m$ ,

$$|(\mathcal{N}(C_i))_i^k| \leq |(\mathcal{N}A)_i^k|.$$

The induction hypothesis implies that for  $l \in \{k-1, k, k+1\}$ ,

$$|\mathcal{N}((C_i)_i^l)| \leq |\mathcal{N}(A_i^l)|.$$

Lemma 5.4 implies that each of the sets  $\mathcal{N}((C_i)_i^{k-1})$ ,  $\mathcal{N}((C_i)_i^k)$ , and  $\mathcal{N}((C_i)_i^{k+1})$  is an initial segment. Hence they are ordered by inclusion, and so applying (5.6) twice shows that there is a number  $l \in \{k-1, k, k+1\}$  such that

$$|(\mathcal{N}(C_i))_i^k| = |\mathcal{N}((C_i)_i^l)| \leq |\mathcal{N}(A_i^l)| \leq |(\mathcal{N}A)_i^k|.$$

This proves the claim.

By Lemma 5.3 and the definition of  $C_i$ , for each  $k \in \mathbb{Z}_m$ , the set

$$\{x \in C_i : x_i = k\}$$

consists precisely of the  $|A_i^k|$  earliest placed elements of  $(\mathbb{Z}_m^n, <_n)$  that satisfy  $x_i = k$ . Thus, if  $A$  is not *i-compressed*, then

$$\sum_{x \in C_i} p_n(x) < \sum_{x \in A} p_n(x),$$

as desired.

Thus, we may assume that  $A$  is *i-compressed* for each  $i \in \{1, \dots, n\}$ . This does not necessarily imply that  $A$  is already an initial segment. For example, the subset

$$\{(0, 0, 0), (0, 0, 1), (0, 1, 0), (1, 0, 0)\} \subseteq \mathbb{Z}_m^3$$

is  $i$ -compressed for  $i = 1, 2, 3$ , but it is not an initial segment.

Lemma 5.3 and the assumption that  $A$  is compressed in each coordinate imply the following crucial fact:

$$(5.8) \quad \text{if } x \notin A, y \in A, \text{ and } x <_n y, \text{ then } x_i \neq y_i \text{ for every } i \in \{1, \dots, n\}.$$

As  $A$  is non-empty and compressed in each coordinate, we may find an index  $i \in \{1, \dots, n\}$  and  $k \in \mathbb{Z}_m$  such that  $A_i^k$  is non-empty and hence contains the vector  $(0, \dots, 0) \in \mathbb{Z}_m^{n-1}$ . This implies that

$$\underbrace{(0, \dots, 0)}_{i-1}, k, \underbrace{(0, \dots, 0)}_{n-i} \in A.$$

Thus, for any  $j \neq i$ , the set  $A_j^0$  is non-empty. Repeating the above argument now shows that  $A$  must contain the vector  $(0, \dots, 0) \in \mathbb{Z}_m^n$ .

Let

$$S_\alpha = \{x \in \mathbb{Z}_m^n : x \leq_n a\}$$

be the longest initial segment contained in  $A$ , and let

$$S_\beta = \{x \in \mathbb{Z}_m^n : s_n(a) \leq x \leq b\}$$

be the first segment in the complement of  $A$ , i.e., the longest segment beginning with  $s_n(a)$  that is entirely contained in  $\mathbb{Z}_m^n \setminus A$ . Finally, let

$$S_\omega = \{x \in \mathbb{Z}_m^n : y \leq_n x \leq_n z\}$$

be the last segment in  $A$ , i.e.,  $S_\omega$  is the longest segment in  $A$  that contains the maximal element of  $A$ , which we have called  $z$ .

As  $A$  is not an initial segment,  $s_n(b) \in A$ . By Proposition 5.2 and the implication (5.8) applied to  $s_n(b)$  and  $b$ , we see that

$$b = (\underbrace{|b|_0, \dots, |b|_0}_{j_b}, s_1(|b|_0), \dots, s_1(|b|_0)) \quad \text{and} \quad s_n(b) = (\underbrace{0, \dots, 0}_{j_b-1}, s_1(|b|_0), 0, \dots, 0).$$

Applying (5.8) to  $s_n(b)$  and  $s_n(a)$  shows that  $s_n(a)_i \neq 0$  for each coordinate  $i \neq j_b$ . Suppose that  $s_n(a)_{j_b} \neq 0$  as well. Then  $s_n(a)$  has no coordinates with value 0, and so Lemma 5.5 yields

$$|\mathcal{N}(S_\alpha \cup \{s_n(a)\}) \setminus \mathcal{N}S_\alpha| = 1.$$

However, Lemma 5.5 also shows that

$$|\mathcal{N}(\{z\}) \setminus \mathcal{N}(\{x \in \mathbb{Z}_m^n : x <_n z\})| \geq 3^k \geq 1,$$

where  $k$  is the number of coordinates of  $z$  with value 0. As  $z$  is the maximal element of  $A$ , these facts imply that

$$|\mathcal{N}((A \cup \{s_n(a)\}) \setminus \{z\})| \leq |\mathcal{N}A|.$$

Moreover, it is clear that  $|(A \cup \{s_n(a)\}) \setminus \{z\}| = |A|$ , and

$$\sum_{x \in (A \cup \{s_n(a)\}) \setminus \{z\}} p_n(x) < \sum_{x \in A} p_n(x),$$

as desired.

Now, suppose that  $s_n(a)_{j_b} = 0$ . From (5.8) applied to  $s_n(b)$  and the elements of  $S_\beta$ , we see that in the case  $j_b \geq 2$ ,

$$v := (\underbrace{|b|_0, \dots, |b|_0}_{j_b-2}, 0, |b|_0, s_1(|b|_0), \dots, s_1(|b|_0)) <_n s_n(a),$$

and in the case  $j_b = 1$ ,

$$v := (|b|_0, \underbrace{s_1(|b|_0), \dots, s_1(|b|_0)}_{n-2}, 0) <_n s_n(a).$$

In either case, using Proposition 5.2 to determine all the vectors in the segment

$$\{x \in \mathbb{Z}_m^n : v <_n x \leq_n b\},$$

we see that the assumption  $s_n(a)_{j_b} = 0$  implies that

$$s_n(a) = (|b|_0, \dots, |b|_0, 0, s_1(|b|_0), \dots, s_1(|b|_0)).$$

$\underbrace{\hspace{10em}}_{j_b-1}$

As a result,

$$S_\beta = \{(\underbrace{|b|_0, \dots, |b|_0}_{j_b-1}, k, s_1(|b|_0), \dots, s_1(|b|_0)) : 0 \leq_1 k \leq_1 |b|_0\},$$

and  $|S_\beta| = p_1(|b|_0)$ . Moreover,  $s_n(a)$  is the only vector in  $S_\beta$  with a coordinate with value 0.

Let  $l = \max\{|S_\beta|, |S_\omega|\}$ , and let  $\tilde{S}_\beta$  be the first  $l$  vectors in  $S_\beta$ . Then by the above statement and Lemma 5.5,

$$|\mathcal{N}(S_\alpha \cup \tilde{S}_\beta) \setminus \mathcal{N}(S_\alpha)| \leq |\tilde{S}_\beta| + 2 = l + 2.$$

Correspondingly, let  $\tilde{S}_\omega$  be the last  $l$  vectors in  $S_\omega$ . We claim that there is at least one vector in  $\tilde{S}_\omega$  that has a coordinate with value 0. Given this claim, Lemma 5.5 implies that

$$|\mathcal{N}(\tilde{S}_\omega) \setminus \mathcal{N}(A \setminus \tilde{S}_\omega)| \geq |\tilde{S}_\omega| + 2 = l + 2,$$

and hence

$$|\mathcal{N}((A \cup \tilde{S}_\beta) \setminus \tilde{S}_\omega)| \leq |\mathcal{N}(A)|,$$

completing the proof as before.

Towards a proof of the claim, we may assume that  $l = |S_\beta| = p_1(|b|_0)$ . Otherwise, the initial element  $y$  of  $S_\omega$  is in  $\tilde{S}_\omega$ , at least  $(n-1)$  of its coordinates have value 0. Indeed, using Proposition 5.2, and applying (5.8) to  $y$  and its immediate predecessor as well as to  $y$  and  $s_n(a)$ , we see that

$$y = (\underbrace{0, \dots, 0}_{j_b-1}, |y|_\infty, 0, \dots, 0).$$

Now, suppose that no vector in  $\tilde{S}_\omega$  has a coordinate with value 0. Then Proposition 5.2 shows there is a fixed coordinate whose value increases successively from one vector to the next in  $\tilde{S}_\omega$ , and the value of this coordinate in the final vector  $z$  of  $\tilde{S}_\omega$  is  $|z|_0$ . Since the value of this coordinate in the first vector in  $\tilde{S}_\omega$  is assumed to be non-zero, we see that

$$p_1(|z|_0) \geq l + 1 = p_1(|b|_0) + 1,$$

and hence that  $|z|_0 \geq s_1(|b|_0)$ . Choose any coordinate  $i \in \{1, \dots, n\}$  that is not  $j_b$ . Such a coordinate exists since  $n \geq 2$ . Then  $y_i = 0$  and  $z_i \geq s_1(|b|_0)$ . Since  $S_\omega$  is a segment and  $b_i \in \{|b|_0, s_1(|b|_0)\}$ , we see from Proposition 5.2 that there is some  $w \in S_\omega$  such that  $w_i = b_i$ , contradicting (5.8).  $\square$

We are now able to prove the isoperimetric inequality on  $R_m^n$  as stated in Section 4.

*Proof of Theorem 4.1.* Let  $A \subset \mathbb{Z}_m^n$  with  $|A| \leq (m-1)^n$ . We may find an integer  $0 \leq k \leq m-2$  such that

$$k^n \leq |A| \leq (k+1)^n.$$

We assert that

$$(5.9) \quad (k+2)^n - k^n \leq |\partial A|.$$

Given this, we see that

$$|\partial A| \geq |A|^{(n-1)/n} \left( \frac{(k+2)^n - k^n}{(k+1)^{n-1}} \right).$$

An elementary calculation shows that for all integers  $k, n \geq 1$ ,

$$\frac{(k+2)^n - k^n}{(k+1)^{n-1}} \geq 2,$$

which yields the desired result.

We now prove (5.9). For an integer  $0 \leq p \leq m^n$ , let  $I_p$  denote the initial segment in  $(\mathbb{Z}_m^n, <_n)$  of length  $p$ . We first claim that for  $0 \leq p \leq m^n - 1$ ,

$$|\partial I_{p+1}| \geq |\partial I_p|$$

Writing  $I_p = \{x \in \mathbb{Z}_m^n : x \leq_n v\}$ , we see from Lemma 5.4 that

$$\partial I_p = \{x \in \mathbb{Z}_m^n : v <_n x \leq v^+\} \quad \text{and} \quad \partial I_{p+1} = \{x \in \mathbb{Z}_m^n : s_n(v) <_n x \leq s_n(v)^+\}.$$

Hence,

$$\partial I_{p+1} = (\partial I_p \cup \{x \in \mathbb{Z}_m^n : v^+ <_n x \leq_n s_n(v)^+\}) \setminus \{s_n(v)\}.$$

Lemma 5.5 shows that the set

$$\{x \in \mathbb{Z}_m^n : v^+ <_n x \leq_n s_n(v)^+\}$$

is not empty, proving the claim.

Let  $S_k$  be the initial segment in  $(\mathbb{Z}_m, <_1)$  of length  $k$ . Proposition 5.2 implies that

$$I_{k^n} = \underbrace{S_k \times \dots \times S_k}_n.$$

Moreover, it follows from the definitions that

$$\mathcal{N}I_{k^n} = \underbrace{\mathcal{N}S_k \times \dots \times \mathcal{N}S_k}_n = \underbrace{S_{k+2} \times \dots \times S_{k+2}}_n = I_{(k+2)^n}.$$

From this, the above claim, and Theorem 5.1, we see that

$$(k+2)^n - k^n \leq |\partial I_{k^n}| \leq |\partial I_{|A|}| \leq |\partial A|,$$

as desired. □

**Corollary 5.6.** *No metric space with at least two points can support a metric cotype 1 inequality.*

*Proof.* It suffices to show that the two point metric space  $\{a, b\}$  with  $d(a, b) = 1$  does not support a metric cotype 1 inequality. Suppose that it does, i.e., there is a constant  $\Gamma \geq 1$  and a scaling function  $m_{1,1}: \mathbb{N} \rightarrow 2\mathbb{N}$  such that the inequality (1.1) holds with  $p = q = 1$ . Fix any  $n \in \mathbb{N}$ , and let  $m = m_{1,1}(n)$ . For any  $k \in \mathbb{N}$ , define  $f^k: \mathbb{Z}_{km}^n \rightarrow \{a, b\}$  by

$$f^k(x) = \begin{cases} a & x \in I_{(km/2-2)^n} \\ b & x \notin I_{(km/2-2)^n}. \end{cases}$$

By [13, Lemma 2.4], we see that for any  $k \in \mathbb{N}$ ,

$$(5.10) \quad \mathbb{E}_{x \in \mathbb{Z}_{km}^n} \sum_{j=1}^n d(f^k(x), f_j^k(x)) \leq \Gamma km \mathbb{E}_{x \in \mathbb{Z}_{km}^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} d(f^k(x), f_\delta^k(x)).$$

From the formula for successors given in Proposition 5.2, we see that the placement of every coordinate of every element in  $I_{(km/2-2)^n}$  is less than  $km/2$ . In other words, every coordinate of every element in  $I_{(km/2-2)^n}$  is contained in the initial segment of  $(\mathbb{Z}_{km}, <_1)$  of length  $km/2 - 1$ . Considering  $\mathbb{Z}_{km}$  as a subset of the unit circle via the map  $z \mapsto e^{2\pi iz/(km)}$ , the initial segment in  $(\mathbb{Z}_{km}, <_1)$  of length  $km/2 - 1$  lies in a sector of angle strictly less than  $\pi$ . However, precomposing with addition by  $km/2$  corresponds to postcomposition by a rotation of angle  $\pi$ . Hence if  $x \in I_{(km/2-2)^n}$ , then  $x + (km/2)\mathbf{e}_j \notin I_{(km/2-2)^n}$  for all  $j = 1, \dots, n$ . Thus, if  $x \in I_{(km/2-2)^n}$ , then  $d(f^k(x), f_j^k(x)) = 1$ . Hence,

$$\mathbb{E}_{x \in \mathbb{Z}_{km}^n} \sum_{j=1}^n d(f^k(x), f_j^k(x)) \geq (km)^{-n} n |I_{(km/2-2)^n}| = (km)^{-n} n (km/2 - 2)^n.$$

On the other hand, as in the proof of Theorem 4.1,

$$|\partial I_{(km/2-2)^n}| = |\mathcal{N} I_{(km/2-2)^n} \setminus I_{(km/2-2)^n}| = |I_{(km/2)^n} \setminus I_{(km/2-2)^n}| = (km/2)^n - (km/2 - 2)^n.$$

This implies that

$$\mathbb{E}_{x \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} d(f(x), f_\delta(x)) = (3km)^{-n} ((km/2)^n - (km/2 - 2)^n).$$

Applying the metric cotype inequality (5.10) and rearranging, we see that

$$n3^n \leq \Gamma km \left( \left( \frac{km}{km-4} \right)^n - 1 \right).$$

Letting  $k$  tend to infinity now yields a contradiction if  $n$  is large enough.  $\square$

## 6. THE INVARIANCE OF METRIC COTYPE

It is clear that metric cotype inequalities are invariant under scaled bi-Lipschitz embeddings. We leave the proof of the following statement to the reader.

**Proposition 6.1.** *Let  $1 \leq p \leq q < \infty$  and  $L \geq 1$ . Suppose  $X$  and  $Y$  are metric spaces such that there is a scaled  $L$ -bi-Lipschitz embedding of  $Y$  into  $X$ . If  $X$  supports a  $(p, q)$  metric cotype inequality with constant  $\Gamma$  and scaling function  $m_{p,q}(n)$ , then  $Y$  supports a  $(p, q)$  metric cotype inequality with the same scaling function  $m_{p,q}(n)$  and constant depending only on  $L$  and  $\Gamma$ .*

A key feature of cotype in the Banach space setting is that it is preserved by a variety of much larger classes of non-linear mappings than just bi-Lipschitz embeddings. The following theorem from [14] is an example of such a result.

**Theorem 6.2** (Naor). *Suppose that  $f: W \hookrightarrow V$  is a quasisymmetric embedding of Banach spaces, and assume that  $V$  has non-trivial type. Then*

$$\begin{aligned} \inf\{q : W \text{ supports a metric cotype } q\text{-inequality}\} \\ \leq \inf\{q : V \text{ supports a metric cotype } q\text{-inequality}\}. \end{aligned}$$

In particular,  $q_W \leq q_V$ .



The proof of Theorem 6.2 relies heavily on the classical Maurey-Pisier theorem [11] and the equivalence between metric and Rademacher cotype. The assumption that  $V$  have non-trivial type is equivalent to the assumption that  $V$  is  $K$ -convex. As shown in [13, Theorem 4.1], this implies that in considering metric cotype  $q$  inequalities on  $V$ , one may assume that  $m_{p,q}(n) = O(n^{1/q})$ . This is also a crucial fact in the proof. Theorem 6.2 and the equivalence between Rademacher and metric cotype immediately yields that  $L_p$  does not quasisymmetrically embed in  $L_2$  if  $p > 2$ . We do not know if Theorem 6.2 is valid in a general setting. This seems to be an interesting and difficult question.

However, the class of scaled snowflaking embeddings, which is smaller than the class of quasisymmetric embeddings, preserves the existence of *some*  $(p, q)$  metric cotype inequality provided that  $m_{p,q}(n) = O(n^{1/q})$ .

**Proposition 6.3.** *Let  $1 \leq p \leq q < \infty$ , and suppose that  $\phi: Y \hookrightarrow X$  is a scaled  $(\alpha, L)$ -snowflaking map. If  $X$  supports a  $(p, q)$  metric cotype inequality with constant  $\Gamma$  and a scaling function satisfying  $m_{p,q}(n) = O(n^{1/q})$ , then  $Y$  supports an  $(\alpha p, q)$  metric cotype inequality with the same scaling function  $m_{p,q}$ , and constant depending only on  $\alpha, L, \Gamma$ , and  $m_{p,q}$ .*

*Proof.* Set  $p' = \alpha p$ . Let  $\Gamma \geq 1$  and  $m_{p,q}$  be the constant and scaling function associated to the  $(p, q)$  metric cotype inequality on  $X$ , respectively. Let  $n \in \mathbb{N}$  and set  $m = m_{p,q}(n)$ , and consider any  $f: \mathbb{Z}_m^n \rightarrow Y$ . Using the fact that  $\phi$  is a  $(\alpha, L)$ -snowflaking map post-composed with a  $c$ -scaling,  $c > 0$ , we see that

$$\begin{aligned} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d_Y(f(\epsilon), f_j(\epsilon))^{p'} &\leq (Lc^{-1})^p \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d_X(\phi \circ f(\epsilon), \phi \circ f_j(\epsilon))^p \\ &\leq (Lc^{-1})^p \Gamma^p m^p n^{1-(p/q)} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1,0,1\}^n} d_X(\phi \circ f_\delta(\epsilon), \phi \circ f(\epsilon))^p \\ &\leq L^{2p} \Gamma^p m^p n^{1-(p/q)} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1,0,1\}^n} d_Y(f(\epsilon), f_\delta(\epsilon))^{p'}. \end{aligned}$$

We see from [13, Lemma 2.3] that  $m \geq \Gamma^{-1} n^{1/q}$ , and by assumption there is a constant  $K \geq 1$  such that  $m \leq K n^{1/q}$ . Thus

$$m^p n^{1-(p/q)} \leq K^p n = K^p n^{p'/q} n^{1-(p'/q)} \leq K^p \Gamma^{p'} m^{p'} n^{1-(p'/q)}.$$

Combining these estimates yields the desired result, with constant  $L^{2p/p'} \Gamma^{(p+p')/p'} K^{p/p'}$  and scaling function  $m_{p,q}$ .  $\square$

*Remark 6.4.* In Theorem 6.2, it is essentially shown that a  $(q, q)$ -metric cotype inequality is preserved under quasisymmetric embeddings. However, even though the class of snowflaking embeddings is much smaller, in Proposition 6.3 the first exponent may change. It is not clear if a better result is possible in this general setting. By using Hölder's inequality and similar estimates for sums, if  $p = q$  in Proposition 6.3, then one can show that  $Y$  supports the following inequality:

$$\mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \left( \sum_{j=1}^n d_Y(f(\epsilon), f_j(\epsilon))^q \right)^\alpha \leq L^2 \Gamma^q m^q \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \left( \mathbb{E}_{\delta \in \{-1,0,1\}^n} d(f(\epsilon), f_\delta(\epsilon))^q \right)^\alpha.$$

It would be interesting to know the consequences of such an inequality.

The following corollary now follows from the fact that a Hilbert space supports a metric cotype 2 inequality with  $m_{2,2} = O(n^{1/2})$  [13, Proposition 3.1].

**Corollary 6.5.** *Suppose that  $Y$  is metric space that  $(\alpha, C)$ -snowflake embeds into a Hilbert space. Then  $Y$  supports a  $(2\alpha, 2)$  metric cotype inequality. In particular,  $q_Y \leq 2$ .*

*Remark 6.6.* A metric space  $Y$  is doubling if there is a number  $N \in \mathbb{N}$  such that for every point  $y \in Y$  and radius  $r > 0$ , the ball  $B(y, r)$  may be covered by at most  $N$  balls of radius  $r/2$ . Assouad's Theorem [6, Theorem 12.2] states that if  $Y$  is doubling, then we may find  $L \geq 1$  and  $n \in \mathbb{N}$ , depending only on  $\alpha$  and the doubling constant of  $Y$ , such that there is an  $(\alpha, L)$ -snowflaking embedding of  $Y$  into  $\mathbb{R}^n$ . Thus Corollary 6.5 implies that for all  $0 < \alpha < 1$ , the space  $Y$  supports a  $(2\alpha, 2)$  metric cotype inequality. We do not know if this implies that  $Y$  supports a metric cotype 2 inequality.

## 7. GROMOV-HAUSDORFF LIMITS

It is not difficult to show that if a metric space  $X$  supports a metric cotype inequality, then the completion  $\overline{X}$  satisfies the same inequality, possibly with a different constant. Essentially the same proof shows that metric cotype inequalities are inherited by Gromov-Hausdorff limits. In this section we use this and related facts to give lower bounds for  $q_X$  in a variety of circumstances. For an introduction to Gromov-Hausdorff limits, see [3].

We begin with some notation. For  $c > 0$ , a (possibly non-continuous) function  $\phi: Y \rightarrow X$  of metric spaces is said to be a  $c$ -rough isometry if for all  $y, z \in Y$ ,

$$|d_Y(y, z) - d_X(\phi(y), \phi(z))| \leq c.$$

A subset  $A$  of a metric space  $X$  is said to be  $c$ -dense if for all  $x \in X$ , there is  $a \in A$  such that  $d_X(a, x) \leq c$ . If the image of a mapping  $\phi: Y \rightarrow X$  is  $c$ -dense in  $X$ , then there is a *rough inverse mapping*  $\tilde{\phi}: X \rightarrow Y$  with the property that  $d_X(\phi \circ \tilde{\phi}(x), x) \leq c$ .

We say that a sequence of compact metric spaces  $\{(X_k, d_k)\}$  Gromov-Hausdorff converges to a metric space  $(X, d)$  if for every  $c > 0$ , there is  $K \in \mathbb{N}$  so that if  $k \geq K$ , then there is a  $c$ -rough isometry from  $X_k$  to  $X$  with  $c$ -dense image.

There is a version of Gromov-Hausdorff convergence that is more appropriate in the non-compact setting. A *pointed metric space* is a pair  $((X, d), p)$  where  $(X, d)$  is a metric space and  $p$  is a point in  $X$ . A sequence of pointed metric spaces  $\{(X_k, d_k), p_k\}$  *pointed Gromov-Hausdorff converges* to a pointed metric space  $((X, d), p)$  if for all  $r > 0$  and all  $c > 0$ , there is  $K \in \mathbb{N}$  such that if  $k \geq K$ , then there is a  $c$ -rough isometry  $\phi: X_k \rightarrow X$  such that  $\phi(X_k)$  is  $c$ -dense in  $B(p, r - c)$ , and  $\phi(p_k) = p$ . This notion restricts to the standard notion in the case that all spaces involved are compact [3, Exercise 8.1.2]

**Proposition 7.1.** *Let  $1 \leq p \leq q < \infty$ . Fix  $\Gamma \geq 1$  and a scaling function  $m_{p,q}: \mathbb{N} \rightarrow 2\mathbb{N}$ . Suppose that each metric space in the sequence of pointed spaces  $\{((X_k, d_k), p_k)\}$  supports a  $(p, q)$  metric cotype inequality with constant  $\Gamma$  with scaling function  $m_{p,q}$ . If  $\{((X_k, d_k), p_k)\}$  Gromov-Hausdorff converges to a pointed space  $((X, d), p_\infty)$ , then  $(X, d)$  supports a  $(p, q)$  metric cotype inequality with constant  $4\Gamma$  and scaling function  $m_{p,q}$ .*

*Proof.* Let  $m = m_{p,q}(n)$ , let  $f: \mathbb{Z}_m^n \rightarrow X$  be a function. Since  $\mathbb{Z}_m^n$  is finite, we may find  $r > 0$  such that  $f(\mathbb{Z}_m^n) \subseteq B_X(p_\infty, r/2)$ . Let  $0 < c < r/2$ . By assumption, there is  $K \in \mathbb{N}$  such that if  $k \geq K$ , then there is a  $c$ -rough isometry  $\phi: X_k \rightarrow X$  such that  $\phi(X_k)$  is  $c$ -dense in  $B(p_\infty, r - c)$ , and hence in  $f(\mathbb{Z}_m^n)$ . As a result there is a  $c$ -rough inverse  $\tilde{\phi}$  of  $\phi$  defined on  $f(\mathbb{Z}_m^n)$ . Thus, for each  $\epsilon \in \mathbb{Z}_m^n$  and  $j = 1, \dots, n$ ,

$$\begin{aligned} |d_X(f(\epsilon), f_j(\epsilon)) - d_X(\phi \circ \tilde{\phi} \circ f(\epsilon), \phi \circ \tilde{\phi} \circ f_j(\epsilon))| &\leq 2c, \text{ and} \\ |d_X(\phi \circ \tilde{\phi} \circ f(\epsilon), \phi \circ \tilde{\phi} \circ f_j(\epsilon)) - d_{X_k}(\tilde{\phi} \circ f(\epsilon), \tilde{\phi} \circ f_j(\epsilon))| &\leq c. \end{aligned}$$

These facts, along with the elementary inequality  $(a + b)^p \leq 2^p(a^p + b^p)$ , where  $a, b \geq 0$ , imply that

$$\mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d_X(f(\epsilon), f_j(\epsilon))^p \leq n(6c)^p + 2^p \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d_{X_k}(\tilde{\phi} \circ f(\epsilon), \tilde{\phi} \circ f_j(\epsilon))^p.$$

We now apply the  $(p, q)$  metric cotype inequality to the map  $\tilde{\phi} \circ f$  and make similar estimates, yielding

$$\begin{aligned} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d_{X_k}(\tilde{\phi} \circ f(\epsilon), \tilde{\phi} \circ f_j(\epsilon))^q &\leq \Gamma^p m^p n^{1-(p/q)} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} d_{X_k}(\tilde{\phi} \circ f(\epsilon), \tilde{\phi} \circ f_\delta(\epsilon))^p \\ &\leq \Gamma^p m^p n^{1-(p/q)} ((6c)^p + 2^p \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} d_X(f(\epsilon), f_\delta(\epsilon))^p). \end{aligned}$$

We now see that

$$\begin{aligned} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d_X(f(\epsilon), f_j(\epsilon))^p &\leq (6c)^p (n + 2^p \Gamma^p m^p n^{1-(p/q)}) \\ &\quad + 4^p \Gamma^p m^p n^{1-(p/q)} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} d_X(f(\epsilon), f_\delta(\epsilon))^p. \end{aligned}$$

Letting  $c$  tend to zero now yields the desired result.  $\square$

*Remark 7.2.* Let  $x_0 \in X$ . The definitions imply that the constant sequence  $\{(X, d), x_0\}_{k=1}^\infty$  pointed Gromov-Hausdorff converges to the completed space  $(\overline{X}, d), x_0$ . For this reason, one usually only considers complete spaces when dealing with Gromov-Hausdorff limits. In any case, Proposition (7.1) implies that if a metric space  $(X, d)$  supports a  $(p, q)$  metric cotype inequality, then the completion  $\overline{X}$  also supports a  $(p, q)$  metric cotype inequality.

A metric space  $(Z, \rho)$  is a *weak tangent* of a metric space  $(X, d)$  at a point  $p \in X$  if there is a sequence  $\{\lambda_k\}$  of positive numbers and a point  $p_0 \in Z$  such that the sequence of pointed metric spaces  $((X, \lambda_k d), p)$  pointed Gromov-Hausdorff converges to  $((Z, \rho), p_0)$ .

**Corollary 7.3.** *Suppose that  $(X, d)$  supports a  $(p, q)$  metric cotype inequality. If  $(Z, \rho)$  is a weak tangent of  $(X, d)$ , then  $(Z, \rho)$  also supports a  $(p, q)$  metric cotype inequality.*

*Proof.* By Proposition 6.1, for any sequence  $\{\lambda_k\}$  of positive numbers, the spaces  $\{(X, \lambda_k d)\}$  support a  $(p, q)$  metric cotype inequality with a uniform constant and scaling function. Proposition 7.1 now yields the desired result.  $\square$

Knowledge of the weak tangents of a space  $X$  can now be used to give lower bounds on  $q_X$ .

*Example 7.4.* Suppose that  $X$  is any one of the spaces  $\mathbb{Q}$ ,  $\mathbb{N}$ , and  $[0, 1]$ , equipped with the standard metric. Then  $X$  has the Hilbert space  $\mathbb{R}$  as a weak tangent, and of course  $X$  is also isometrically embedded in  $\mathbb{R}$ . Since no Banach space has Rademacher cotype less than 2, and every Hilbert space has Rademacher cotype 2, Theorem 1.2 and Corollary 7.3 show that  $q_X = 2$ .

*Example 7.5.* Suppose that a metric space  $(X, d)$  has a weak tangent that contains a rectifiable curve. Since a rectifiable curve is a bi-Lipschitz image of  $[0, 1]$ , we may conclude from the Example 7.4 and Corollary 7.3 that  $q_X \geq 2$ .

## 8. LINE FITTING

In this section we prove Theorems 1.4 and 1.5 using the ideas of the previous sections and the characterization of snowflake spaces, due to Laakso, that is found in the appendix to [18].

**Definition 8.1** (Laakso). *A metric space  $(X, d)$  is line fitting if for every  $c > 0$ , there is a metric  $d_c$  on the disjoint union  $X \coprod [0, 1]$  with the following properties:*

- *there is a number  $\lambda_c > 0$  such that  $d_c|_{X \times X} = \lambda_c d$ ,*
- *$d_c|_{[0,1] \times [0,1]}$  is the standard metric on  $[0, 1]$ ,*
- *for each point  $t \in [0, 1]$ , there is a point  $x_t \in X$  such that  $d_c(x_t, t) < c$ .*

**Theorem 8.2** (Laakso). *A metric space is a 1-snowflake if and only if it is line fitting.*

Given this result, the proof of Theorem 1.4 becomes very similar to the proof of Proposition 7.1.

*Proof of Theorem 1.4.* Suppose that  $(X, d)$  is a 1-snowflake. Let  $1 \leq p \leq q < 2$ , and suppose that  $(X, d)$  supports a  $(p, q)$  metric cotype inequality with constant  $\Gamma \geq 1$  and scaling function  $m_{p,q}(n)$ . We will show that  $[0, 1]$  also supports such an inequality. This is a contradiction to Corollary 7.3, as discussed in example 7.4.

Let  $n \in \mathbb{N}$ , let  $m = m_{p,q}(n) \in 2\mathbb{N}$ , and consider  $f: \mathbb{Z}_m^n \rightarrow [0, 1]$ . Fix  $c > 0$ . By Theorem 8.2, we may find a metric  $d_c$  on  $X \coprod [0, 1]$  as in Definition 8.1. Since  $[0, 1]$  is contained in  $\mathcal{N}_{d_c}(X, c)$ , we may find a function  $\tilde{f}: \mathbb{Z}_m^n \rightarrow X$  with the property that for any  $\epsilon \in \mathbb{Z}_m^n$ ,

$$d_c(\tilde{f}(\epsilon), f(\epsilon)) < c.$$

From this, the triangle inequality, and the other properties of  $d_c$ , we see that

$$\mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n |f(\epsilon) - f_j(\epsilon)|^p \leq n(4c)^p + 2^p \lambda_c^p \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n d(\tilde{f}(\epsilon), \tilde{f}_j(\epsilon))^p.$$

A similar argument shows that

$$\mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} d(\tilde{f}(\epsilon), \tilde{f}_\delta(\epsilon))^p \leq \lambda_c^{-p} (4c)^p + 2^p \lambda_c^{-p} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} |f(\epsilon) - f_\delta(\epsilon)|^p.$$

Applying the  $(p, q)$  metric cotype inequality on  $X$  to the function  $\tilde{f}$  and combining with the previous estimates produces

$$\begin{aligned} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \sum_{j=1}^n |f(\epsilon) - f_j(\epsilon)|^p &\leq (4c)^p (n + 2^p \Gamma^p m^p n^{1-(p/q)}) \\ &\quad + 4^p \Gamma^p m^p n^{1-(p/q)} \mathbb{E}_{\epsilon \in \mathbb{Z}_m^n} \mathbb{E}_{\delta \in \{-1, 0, 1\}^n} |f(\epsilon) - f_\delta(\epsilon)|^p. \end{aligned}$$

Letting  $c$  tend to 0 completes the proof. □

*Proof of Theorem 1.5.* The condition that  $(X, d)$  is an  $s$ -snowflake implies that there is an  $(s, 1)$ -snowflake embedding from a 1-snowflake space into  $(X, d)$ . Thus the desired result follows from Theorem 1.4 and Proposition 6.3. □

## REFERENCES

- [1] Yair Bartal, Nathan Linial, Manor Mendel, and Assaf Naor. On metric Ramsey-type phenomena. *Ann. of Math. (2)*, 162(2):643–709, 2005.
- [2] Béla Bollobás and Imre Leader. An isoperimetric inequality on the discrete torus. *SIAM J. Discrete Math.*, 3(1):32–37, 1990.

- [3] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
- [4] Jeff Cheeger, Bruce Kleiner, and Assaf Naor. Compression bounds for Lipschitz maps from the Heisenberg group to  $l_1$ . (preprint).
- [5] Guy David and Stephen Semmes. *Fractured fractals and broken dreams*, volume 7 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1997. Self-similar geometry through metric and measure.
- [6] Juha Heinonen. *Lectures on analysis on metric spaces*. Universitext. Springer-Verlag, New York, 2001.
- [7] Nigel J. Kalton. The nonlinear geometry of Banach spaces. *Rev. Mat. Complut.*, 21(1):7–60, 2008.
- [8] Leonid V. Kovalev. Conformal dimension does not assume values between zero and one. *Duke Math. J.*, 134(1):1–13, 2006.
- [9] Imre Leader. Discrete isoperimetric inequalities. In *Probabilistic combinatorics and its applications (San Francisco, CA, 1991)*, volume 44 of *Proc. Sympos. Appl. Math.*, pages 57–80. Amer. Math. Soc., Providence, RI, 1991.
- [10] Bernard Maurey. Type, cotype and  $K$ -convexity. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1299–1332. North-Holland, Amsterdam, 2003.
- [11] Bernard Maurey and Gilles Pisier. Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach. *Studia Math.*, 58(1):45–90, 1976.
- [12] Manor Mendel and Assaf Naor. Scaled Enflo type is equivalent to Rademacher type. *Bull. Lond. Math. Soc.*, 39(3):493–498, 2007.
- [13] Manor Mendel and Assaf Naor. Metric cotype. *Ann. of Math. (2)*, 168(1):247–298, 2008.
- [14] Assaf Naor. An application of metric cotype to quasisymmetric embeddings. available at <http://www.cims.nyu.edu/naor/>.
- [15] Assaf Naor, Yuval Peres, Oded Schramm, and Scott Sheffield. Markov chains in smooth Banach spaces and Gromov-hyperbolic metric spaces. *Duke Math. J.*, 134(1):165–197, 2006.
- [16] W. H. Schikhof. *Ultrametric calculus*, volume 4 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1984. An introduction to  $p$ -adic analysis.
- [17] A. F. Timan and I. A. Vestfrid. Any separable ultrametric space is isometrically embeddable in  $l_2$ . *Funktional. Anal. i Prilozhen.*, 17(1):85–86, 1983.
- [18] Jeremy T. Tyson and Jang-Mei Wu. Characterizations of snowflake metric spaces. *Ann. Acad. Sci. Fenn. Math.*, 30(2):313–336, 2005.
- [19] M. S. Waterman, T. F. Smith, M. Singh, and W. A. Beyer. Additive evolutionary trees. *J. Theoret. Biol.*, 64(2):199–213, 1977.

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