

MODULUS AND POINCARÉ INEQUALITIES ON NON-SELF-SIMILAR SIERPIŃSKI CARPETS

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ABSTRACT. A carpet is a metric space homeomorphic to the Sierpiński carpet. We characterize, within a certain class of examples, non-self-similar carpets supporting curve families of nontrivial modulus and supporting Poincaré inequalities. Our results yield new examples of compact doubling metric measure spaces supporting Poincaré inequalities: these examples have no manifold points, yet embed isometrically as subsets of Euclidean space.

1. INTRODUCTION

Metric spaces equipped with doubling measures that support Poincaré inequalities (also known as *PI spaces*) are ideal environments for first-order analysis and differential geometry [19], [10], [20], [23], [25]. Extending the scope of this theory by verifying Poincaré inequalities on new classes of spaces is a problem of high interest and relevance. Previously, several classes of spaces have been shown to support Poincaré inequalities:

- compact Riemannian manifolds or noncompact Riemannian manifolds satisfying suitable curvature bounds [9],
- Carnot groups and more general sub-Riemannian manifolds equipped with Carnot-Carathéodory (CC) metric [19], [18], [21],
- boundaries of certain hyperbolic Fuchsian buildings, see Bourdon and Pajot [6],
- Laakso’s spaces [28],
- linearly locally contractible manifolds with good volume growth [32].

These examples fall into two (overlapping) classes: examples for which the underlying topological space is a manifold, and abstract metric examples which admit no bi-Lipschitz embedding into any finite-dimensional Euclidean space. Such bi-Lipschitz nonembeddability follows from Cheeger’s celebrated Rademacher-style differentiation theorem in PI spaces, as explained in [10, §14]. Euclidean bi-Lipschitz nonembeddability is known, for instance, for all nonabelian Carnot groups and other regular sub-Riemannian manifolds, as well as for the examples of Bourdon and Pajot [6] and Laakso [28].

Date: February 2, 2013.

2010 Mathematics Subject Classification. 30L99; 31E05; 28A80.

Key words and phrases. Sierpiński carpet, doubling measure, modulus, Poincaré inequality, Gromov–Hausdorff tangent cone.

JMM and JTT were supported by US National Science Foundation Grant DMS-0901620, JMM was supported by EPSRC grant “Geometric and analytic aspects of infinite groups”, JTT was supported by US National Science Foundation Grant DMS-1201875, KW supported by Academy of Finland Grants 120972 and 128144, the Swiss National Science Foundation, ERC Project CG-DICE, and European Science Council Project HCAA.

The preceding dichotomy should not be taken too seriously. Nonabelian Carnot groups equipped with the CC metric, for instance, have underlying space which is a topological manifold, yet do not admit any Euclidean bi-Lipschitz embedding. On the other hand, it is certainly possible to construct Euclidean subsets with some nonmanifold points which are PI spaces. This can be done, for instance, by appealing to various gluing theorems for PI spaces, see [19, Theorem 6.15] (reproduced as Theorem 2.2 in this paper) for a general result along these lines. However, the following question appears to have been unaddressed in the literature until now.

Question 1.1. Do there exist sets $X \subset \mathbb{R}^N$ (for some N) with **no** manifold points which are PI spaces when equipped with the Euclidean metric and some suitable measure?

In connection with Question 1.1 we recall the examples constructed by Heinonen and Hanson [16]. For each $n \geq 2$, these authors construct a compact, geodesic, Ahlfors n -regular PI space of topological dimension n with no manifold points. They suggest [16, p. 3380], but do not check, that their nonmanifold example admits a bi-Lipschitz embedding into some Euclidean space. Note that the question about embeddability of the Heinonen–Hansen example is not resolved by Cheeger’s work, since this example admits almost everywhere unique tangent cones coinciding with \mathbb{R}^n .

The examples of PI spaces due to Bourdon and Pajot [6] comprise a class of compact metric spaces arising as the Gromov boundaries of certain hyperbolic groups acting geometrically on Fuchsian buildings. Topologically, all of the Bourdon–Pajot examples are homeomorphic to the Menger sponge. It is well-known that ‘typical’ Gromov hyperbolic groups have Menger sponge boundaries. While examples of Gromov hyperbolic groups with Sierpiński carpet boundary do exist, it is not presently known whether any such boundary can verify a Poincaré inequality in the sense of Heinonen and Koskela.

Question 1.2. Do there exist PI spaces that are homeomorphic to the Sierpiński carpet?

In this paper we answer Questions 1.1 and 1.2 affirmatively. We identify a new class of doubling metric measure spaces supporting Poincaré inequalities. Our main results are Theorem 1.5 and 1.6. Our spaces have no manifold points, indeed, they are all homeomorphic to the Sierpiński carpet. On the other hand, all of our examples arise as explicit subsets of the plane equipped with the Euclidean metric and the Lebesgue measure. These are the first examples of compact subsets of Euclidean space without interior that support Poincaré inequalities for the usual Lebesgue measure.

To fix notation and terminology we recall the notion of Poincaré inequality on a metric measure space as introduced by Heinonen and Koskela [19]. Let (X, d, μ) be a metric measure space, i.e., (X, d) is a metric space and μ is a Borel measure which assigns positive and finite measure to all open balls in X . A Borel function $\rho : X \rightarrow [0, \infty]$ is an *upper gradient* of a function $u : X \rightarrow \mathbb{R}$ if $|u(x) - u(y)| \leq \int_\gamma \rho \, ds$ whenever γ is a rectifiable curve joining x to y .

Definition 1.3 (Heinonen–Koskela). Fix $p \geq 1$. The space (X, d, μ) is said to support a *p -Poincaré inequality* if there exist constants $C, \lambda \geq 1$ so that for any continuous function $u : X \rightarrow \mathbb{R}$ with upper gradient $\rho : X \rightarrow [0, \infty]$, the inequality

$$(1.1) \quad \int_B \left| u - \int_B u \, d\mu \right| d\mu \leq C \operatorname{diam}(B) \left(\int_{\lambda B} \rho^p \, d\mu \right)^{1/p}$$

holds for every ball $B = B(x, r) \subset X$. Here we denote, for a subset $E \subset X$ of positive measure, the mean value of a function $u : E \rightarrow \mathbb{R}$ by $\int_E u \, d\mu = \frac{1}{\mu(E)} \int_E u \, d\mu$.

The validity of a Poincaré inequality in the sense of Definition 1.3 reflects strong connectivity properties of the underlying space. Roughly speaking, metric measure spaces (X, d, μ) supporting a Poincaré inequality have the property that any two regions are connected by a rich family of relatively short curves which are evenly distributed with respect to the background measure μ . (For a more precise version of this statement, see Theorem 2.1.) The main results of this paper are a reflection and substantiation of this general principle in the setting of a highly concrete collection of planar examples.

We now turn to a description of those examples. To each sequence $\mathbf{a} = (a_1, a_2, \dots)$ consisting of reciprocals of odd integers strictly greater than one we associate a modified Sierpiński carpet $S_{\mathbf{a}}$ by the following procedure. Let $T_0 = [0, 1]^2$ be the unit square and let $S_{\mathbf{a},0} = T_0$. Consider the standard tiling of T_0 by essentially disjoint closed congruent subsquares of side length a_1 . Let \mathcal{T}_1 denote the family of such subsquares obtained by deleting the central (concentric) subsquare, and let $S_{\mathbf{a},1} = \cup\{T : T \in \mathcal{T}_1\}$. Again, let \mathcal{T}_2 denote the family of essentially disjoint closed congruent subsquares of each of the elements of \mathcal{T}_1 with side length $a_1 a_2$ obtained by deleting the central (concentric) subsquare from each square in \mathcal{T}_1 , and let $S_{\mathbf{a},2} = \cup\{T : T \in \mathcal{T}_2\}$. Continuing this process, we construct a decreasing sequence of compact sets $\{S_{\mathbf{a},m}\}_{m \geq 0}$ and an associated carpet

$$S_{\mathbf{a}} := \bigcap_{m \geq 0} S_{\mathbf{a},m}.$$

For example, when $\mathbf{a} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots)$, the set $S_{\mathbf{a}}$ is the classical Sierpiński carpet $S_{1/3}$ (Figure 1). For any \mathbf{a} , $S_{\mathbf{a}}$ is a compact, connected, locally connected subset of the plane without interior and with no local cut points. By a standard fact from topology, $S_{\mathbf{a}}$ is homeomorphic to the Sierpiński carpet $S_{1/3}$.

For each $k \in \mathbb{N}$, we will denote by $S_{1/(2k+1)}$ the self-similar carpet $S_{\mathbf{a}}$ associated to the constant sequence $\mathbf{a} = (\frac{1}{2k+1}, \frac{1}{2k+1}, \frac{1}{2k+1}, \dots)$. For each k , the carpet $S_{1/(2k+1)}$ has Hausdorff dimension equal to

$$(1.2) \quad Q_k = \frac{\log((2k+1)^2 - 1)}{\log(2k+1)} = \frac{\log(4k^2 + 4k)}{\log(2k+1)} < 2$$

and is Ahlfors regular in that dimension.

The starting point for our investigations was the following well-known fact.

Proposition 1.4. *For each k , the carpet $S_{1/(2k+1)}$, equipped with Euclidean metric and Hausdorff measure in its dimension Q_k , does **not** support any Poincaré inequality.*

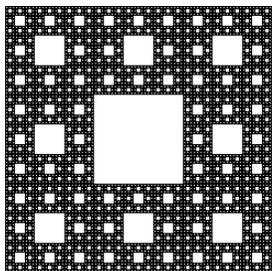


FIGURE 1. $S_{1/3}$

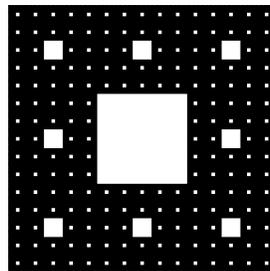


FIGURE 2. $S_{(1/3, 1/5, 1/7, \dots)}$

Several proofs for Proposition 1.4 can be found in the literature. Bourdon and Pajot [7] provide an elegant argument involving the mutual singularity of one-dimensional Lebesgue measure and the push forward of the Q_k -dimensional Hausdorff measure on $S_{1/(2k+1)}$ under projection to a coordinate axis. A different argument involving modulus computations can be found in the monograph by the first two authors [29].

In this paper, we study non-self-similar carpets $S_{\mathbf{a}}$ for which \mathbf{a} is not a constant sequence. We are primarily interested in the case when $S_{\mathbf{a}}$ has Hausdorff dimension two. It is easy to see that this holds, for instance, if the sequence (a_m) of scaling ratios tends to zero, i.e., $\mathbf{a} \in c_0$. Figure 2 illustrates the set $S_{(1/3, 1/5, 1/7, \dots)}$.

Note that the left and right hand edges of $S_{\mathbf{a}}$ are separated by the generalized Cantor set

$$C_{\mathbf{a}} := S_{\mathbf{a}} \cap \left(\left\{ \frac{1}{2} \right\} \times [0, 1] \right).$$

This Cantor set will have positive length if and only if the length at each stage, $\prod_{j=1}^m (1 - a_j)$, remains bounded away from zero. After taking logarithms, this is seen to be equivalent to $\mathbf{a} \in \ell^1$.

In a similar fashion, we see that $\text{Area}(S_{\mathbf{a}}) = \mathcal{H}^2(S_{\mathbf{a}})$ is positive if and only if $\text{Area}(S_{\mathbf{a},m}) = \prod_{j=1}^m (1 - a_j^2)$ is bounded away from zero, i.e., $\mathbf{a} \in \ell^2$.

We equip $S_{\mathbf{a}}$ with the Euclidean metric d and the canonically defined measure μ arising as the weak limit of normalized Lebesgue measures on the precarpet $S_{\mathbf{a},m}$. For all \mathbf{a} , the measure μ is doubling. Under the assumption $\mathbf{a} \in \ell^2$, μ is Ahlfors 2-regular and is comparable (with constant depending only on $\|\mathbf{a}\|_2$) to the restriction of Lebesgue measure to $S_{\mathbf{a}}$. For these and other facts, see Proposition 3.1.

We now state our main theorems.

Theorem 1.5. *The carpet $(S_{\mathbf{a}}, d, \mu)$ supports a 1-Poincaré inequality if and only if $\mathbf{a} \in \ell^1$.*

Under the assumption of Theorem 1.5, the 1-modulus of all horizontal paths in $S_{\mathbf{a}}$ is easily seen to be positive. This fact follows from the usual Fubini argument, since the cut set $C_{\mathbf{a}}$ has positive length. The difficult part of the proof of Theorem 1.5 is the verification of the 1-Poincaré inequality. This is done using a theorem of Keith (Theorem 2.1) and a combinatorial procedure involving concatenation of curve families of positive 1-modulus.

Theorem 1.6. *The following are equivalent:*

- (a) $(S_{\mathbf{a}}, d, \mu)$ supports a p -Poincaré inequality for each $p > 1$,
- (b) $(S_{\mathbf{a}}, d, \mu)$ supports a p -Poincaré inequality for some $p > 1$,
- (c) $\mathbf{a} \in \ell^2$.

For $\mathbf{a} \in \ell^2 \setminus \ell^1$, the p -modulus of all horizontal paths in $S_{\mathbf{a}}$ is equal to zero for any p . However, the p -modulus ($p > 1$) of all rectifiable paths is positive. In section 6 we exhibit explicit path families with positive modulus. This provides a first step towards our eventual verification of the Poincaré inequality. Such verification in this context relies on the same theorem of Keith and a similar concatenation argument, starting from curve families of positive p -modulus as constructed above.

It is not unexpected, and seems to have been informally recognized, that a generalized Sierpiński carpet $S_{\mathbf{a}}$ admits some Poincaré inequalities, provided the sequence \mathbf{a} tends to zero sufficiently rapidly. Indeed, if \mathbf{a} tends rapidly to zero then the omitted squares at each stage of the construction occupy a vanishingly small proportion of their parent square; this leaves plenty of room in

the complementary region to construct well distributed curve families. The essential novelty of Theorems 1.5 and 1.6 lies in their sharp character; we identify the precise summability conditions necessary and sufficient for the validity of the p -Poincaré inequality for each choice of $p \in [1, \infty)$. Note that, by Theorems 1.5 and 1.6, if $\mathbf{a} \in \ell^2 \setminus \ell^1$, then $(S_{\mathbf{a}}, d, \mu)$ supports a p -Poincaré inequality for each $p > 1$, but does not support a 1-Poincaré inequality. A significant recent result of Keith and Zhong [25] asserts that the set of values of p for which a given complete PI space supports a p -Poincaré inequality, is necessarily a relatively open subset of $[1, +\infty)$.

Remarkably, the ℓ^2 summability condition on the defining sequence \mathbf{a} has recently arisen in a rather different (although related) context. To wit, Doré and Maleva [11] show that when $\mathbf{a} \in c_0 \setminus \ell^2$, the compact set $S_{\mathbf{a}}$ is a *universal differentiability set*, i.e., it contains a differentiability point for every real-valued Lipschitz function on \mathbb{R}^2 .

Theorems 1.5 and 1.6 have a number of interesting consequences which we now enunciate.

Corollary 1.7. *There exist compact planar sets of topological dimension one that are Ahlfors 2-regular and 2-Loewner when equipped with the Euclidean metric and the Lebesgue measure.*

For each $\mathbf{a} \in \ell^2$, the carpet $S_{\mathbf{a}}$ verifies the conditions in Corollary 1.7. This follows from Theorem 1.6 and the equivalence of the Q -Loewner condition with the Q -Poincaré inequality in quasiconvex Ahlfors Q -regular spaces [19]. We remark that the examples of Bourdon–Pajot [6] and Laakso [28] are Q -regular Q -Loewner metric spaces of topological dimension one, however, these examples admit no bi-Lipschitz embedding into any finite-dimensional Euclidean space.

Corollary 1.8. *There exists a compact set $S \subset \mathbb{R}^2$, equipped with the Euclidean metric and a doubling measure, with the following properties: S supports no p -Poincaré inequality for any finite p , yet every strict weak tangent of S supports a 1-Poincaré inequality with universal constants. Moreover, S can be chosen to be quasiconvex and uniformly locally Gromov–Hausdorff close to planar domains.*

It is a general principle of analysis in metric spaces that quantitative geometric or analytic conditions often persist under Gromov–Hausdorff convergence. In particular, quantitative and scale-invariant conditions pass to weak tangent spaces. For instance, every weak tangent of a given doubling metric measure space satisfying a p -Poincaré inequality is again doubling and satisfies the same p -Poincaré inequality (see Theorem 2.5 for a version of this result used in this paper). Corollary 1.8 shows that weak tangent spaces can be significantly better behaved than the spaces from which they are derived, even in the presence of other good geometric properties.

The indicated example can be obtained by choosing $S = S_{\mathbf{a}}$ for any $\mathbf{a} \in c_0 \setminus \ell^2$. This follows from Theorem 1.6 and Proposition 4.4 discussed in section 4.1, where further details of the proof of Corollary 1.8 can be found.

A *carpet* is a metric measure space homeomorphic to $S_{1/3}$. There has been considerable interest of late in the problem of quasisymmetric uniformization of carpets by either round carpets or slit carpets [3], [4], [5], [31], [30]. The following results are additional consequences of Theorem 1.6.

Corollary 1.9. *There exist round carpets in \mathbb{R}^2 which are Ahlfors 2-regular and support a p -Poincaré inequality for some $p < 2$.*

Corollary 1.10. *There exist parallel slit carpets which are Ahlfors 2-regular and support a p -Poincaré inequality for some $p < 2$.*

Recall that a planar carpet is said to be a *round carpet* if all of its peripheral circles are round geometric circles. A *slit carpet* is a carpet which is a Gromov–Hausdorff limit of a sequence of planar slit domains equipped with the internal metric. Recall that a domain $D \subset \mathbb{C}$ is a *slit domain* if $D = D' \setminus \bigcup_{i \in I} \gamma_i$, where D' is a simply connected domain and $\{\gamma_i\}_{i \in I}$ is a collection (of arbitrary cardinality) of disjoint closed arcs contained in D' . We admit the possibility that some of these arcs are degenerate, i.e., reduce to a point. A slit domain, resp. a slit carpet, is *parallel* if the nondegenerate arcs are parallel line segments, resp. if it is a limit of parallel slit domains.

Corollaries 1.9 and 1.10 are proved in section 7. Corollary 1.9 follows from Theorem 1.6 and results of Bonk and Koskela–MacManus on quasimetric uniformization of carpets and quasimetric invariance of Poincaré inequalities on Ahlfors regular spaces. Corollary 1.10 follows from Theorem 1.6, Koebe’s uniformization theorem and the same work of Koskela–MacManus. Indeed, every carpet $S_{\mathbf{a}}$ with $\mathbf{a} \in \ell^2$ is quasimetrically equivalent to both a round carpet and also to a slit carpet with the stated properties.

1.1. Outline of the paper. In section 2 we recall general facts about analysis in metric spaces, particularly, facts about Poincaré inequalities in the sense of Definition 1.3. In section 3 we prove basic metric and measure-theoretic properties of the carpets $S_{\mathbf{a}}$. In particular, we show that the canonical measure on $S_{\mathbf{a}}$ is always a doubling measure, and we indicate in which situations it verifies upper or lower mass bounds.

Section 4 is devoted to the necessity of the ℓ^2 summability condition for the validity of Poincaré inequalities on the carpets $S_{\mathbf{a}}$. The main result of this section, Proposition 4.2, shows that ℓ^2 summability of \mathbf{a} is best possible for such conclusions. We also describe in more detail the weak tangents of the carpets $S_{\mathbf{a}}$ and substantiate Corollary 1.8.

Our proofs of the sufficiency of the summability criteria in Theorems 1.5 and 1.6 are contained in sections 5 and 6, respectively. In the setting of Theorem 1.5, where $\mathbf{a} \in \ell^1$, the Cantor set corresponding to the thinnest part of the carpet has positive length. This enables us to give a combinatorial construction of parameterized curve families that join arbitrary pairs of points in $S_{\mathbf{a}}$ and verifies a modulus lower estimate due to Keith (Theorem 2.1) known to be equivalent to the Poincaré inequality in a wide setting.

In the setting of Theorem 1.6, where \mathbf{a} is only assumed to be in ℓ^2 , a different technique is required. The key step is to perform, in the special case of the carpets $S_{\mathbf{a}}$, the following abstract procedure: in a metric space (X, d) endowed with a wide supply of rectifiable curves (in our case, \mathbb{R}^2), deform a given curve family so as to avoid a prespecified obstacle, at a small quantitative multiplicative cost to the p -modulus. Iterating this procedure produces curve families of positive p -modulus that avoid a countable family of obstacles of prespecified geometric sizes. Our implementation, while not completely general, covers a wider class of residual sets than just carpets: see Theorem 6.7 for a precise statement.

In both cases, our proof of the suitable Poincaré inequalities makes substantial use of the precise rectilinear structure of carpets. Hence, the validity of a Poincaré inequality on the more general class of residual sets indicated in the preceding paragraph is less clear.

In the final section (Section 7) we discuss uniformization of the carpets $S_{\mathbf{a}}$ by either round carpets or slit carpets. In particular, we establish Corollaries 1.9 and 1.10.

1.2. Acknowledgements. We are grateful to Mario Bonk for numerous discussions and especially for suggestions concerning uniformization of metric carpets. We also thank Jasun Gong and Hrant

Hakobyan for very helpful remarks. We wish to extend particular thanks to the referee for an extremely careful reading of the paper and for his or her detailed and constructive input.

Research for this paper was performed while the first and third authors were at the University of Illinois and during visits of all three authors to the Mathematics Institute at the University of Bern. The hospitality of both institutions is gratefully appreciated.

2. PRELIMINARIES

2.1. Basic definitions and notation. If $B = B(x, r)$ denotes a ball in a metric space $X = (X, d)$, we write λB for the dilated ball $B(x, \lambda r)$.

A *metric measure space* is a metric space (X, d) equipped with a Borel measure μ that is finite and positive on balls. The measure μ is *doubling* if there exists a constant $C > 0$ so that $\mu(B(x, 2r)) \leq C\mu(B(x, r))$ for all metric balls $B(x, r)$ in X . It is *Ahlfors Q -regular* for some $Q > 0$ if there exists a constant $C > 0$ so that $r^Q/C \leq \mu(B(x, r)) \leq Cr^Q$ for all metric balls $B(x, r)$ in X with $0 < r < \text{diam } X$. We say that μ is *Ahlfors regular* if it is Ahlfors Q -regular for some $Q > 0$. It is well known that any Ahlfors Q -regular measure on a metric space is comparable to the Hausdorff Q -measure \mathcal{H}^Q , and hence that \mathcal{H}^Q is also Ahlfors Q -regular in that case. Ahlfors regular measures are always doubling. Let us remark that we always denote by \mathcal{H}^s the s -dimensional Hausdorff measure in any metric space; we normalize these measures so that \mathcal{H}^n coincides with Lebesgue measure in \mathbb{R}^n .

A metric space (X, d) is said to be *quasiconvex* if there exists a constant C so that any pair of points $x, y \in X$ can be joined by a rectifiable path γ whose length is no more than $Cd(x, y)$. A metric space is quasiconvex if and only if it is bi-Lipschitz equivalent to a length metric space.

Every doubling metric measure space admitting a Poincaré inequality is quasiconvex, see for instance [15] or [10]. By making use of quasiconvexity, we may assume that $\lambda = 1$ in (1.1), at the cost of increasing the value of C [15, Corollary 9.8].

2.2. Poincaré inequalities and moduli of curve families. The following result of Keith [22, Theorem 2] will be of great importance in this paper.

Theorem 2.1 (Keith). *Fix $p \geq 1$. Let (X, d, μ) be a complete, doubling metric measure space. Then X admits a p -Poincaré inequality if and only if there exist constants $C_1 > 0$ and $C_2 \geq 1$ so that*

$$(2.1) \quad d(x, y)^{1-p} \leq C_1 \text{mod}_p(\Gamma_{xy}; \mu_{xy}^{C_2})$$

for every pair of distinct points $x, y \in X$.

Here $\text{mod}_p(\Gamma_{xy}; \mu_{xy}^C)$ denotes the p -modulus of the curve family Γ_{xy} joining x to y , where the measure μ_{xy}^C is the symmetric Riesz kernel

$$\mu_{xy}^C(A) = \int_{A \cap B_{xy}^C} \frac{d(x, z)}{\mu(B(x, d(x, z)))} + \frac{d(y, z)}{\mu(B(y, d(y, z)))} d\mu(z),$$

where $B_{xy}^C = B(x, Cd(x, y)) \cup B(y, Cd(x, y))$. We recall that

$$\text{mod}_p(\Gamma; \nu) := \inf \int \rho^p d\nu$$

for a Borel measure ν on (X, d) . Here the infimum is taken over all nonnegative Borel functions ρ which are *admissible* for Γ , i.e., for which $\int_\gamma \rho ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. When (X, d) is endowed with a fixed ambient measure μ , we abbreviate $\text{mod}_p \Gamma = \text{mod}_p(\Gamma; \mu)$.

2.3. Poincaré inequalities and metric gluings. The Poincaré inequality (1.1) is maintained under metric gluings. The following is a special case of a more general theorem of Heinonen and Koskela [19, Theorem 6.15], see also [16, Theorem 3.3].

Theorem 2.2 (Heinonen–Koskela). *Let X and Y be locally compact Ahlfors Q -regular metric measure spaces, $Q > 1$, let $A \subset X$ be a closed subset, and let $\iota : A \rightarrow Y$ be an isometric embedding. Let $p > 1$. Assume that both X and Y support a p -Poincaré inequality and that the inequality*

$$\min \{ \mathcal{H}_\infty^{Q-1}(A \cap B_X(x, r)), \mathcal{H}_\infty^{Q-1}(\iota(A) \cap B_Y(y, r)) \} \geq cr^{Q-1}$$

holds for all $x \in A$, $y \in \iota(A)$ and $0 < r < \min\{\text{diam } X, \text{diam } Y\}$, where the constant $c > 0$ is independent of x , y and r . Then $X \cup_A Y$ supports a p -Poincaré inequality. The data for the p -Poincaré inequality on $X \cup_A Y$ depends quantitatively on the Ahlfors regularity and Poincaré inequality data of X and Y , on p , and on the above constant c .

We recall that the metric gluing $X \cup_A Y$ is the quotient space obtained by imposing on the disjoint union $X \coprod Y$ the equivalence relation which identifies each $a \in A$ with its image $\iota(a)$. We equip this space with a natural metric which extends the metrics on X and Y as follows: for points $x \in X$ and $y \in Y$, let $d(x, y) = \inf\{d(x, a) + d(\iota(a), y) : a \in A\}$. Observe that the Q -regular measures on X and Y , respectively, combine to give a measure on $X \cup_A Y$ which is also Q -regular.

2.4. Gromov–Hausdorff convergence and weak tangents. A metric space (X, d) is *proper* if closed and bounded sets are compact.

Definition 2.3. A sequence of pointed proper metric measure spaces

$$\{(X_n, x_n, d_n, \mu_n)\}$$

converges to a pointed metric measure space (X, x, d, μ) if there exists a pointed proper metric space (Z, z, ρ) and isometric embeddings $f_n : X_n \rightarrow Z$, $f : X \rightarrow Z$ so that $f_n(x_n) = f(x) = z$ for all n , $(f_n)_\# \mu_n \rightarrow f_\# \mu$ weakly, and $f_n(X_n) \rightarrow f(X)$ in the following sense: for all $R > 0, \epsilon > 0$ there exists N so that for all $n \geq N$, $f_n(X_n) \cap B(z, R)$ is contained in the ϵ -neighborhood of $f(X)$, and $f(X) \cap B(z, R)$ is contained in the ϵ -neighborhood of $f_n(X_n)$.

We emphasize that the spaces X_n, X are not assumed to be compact. For the notion of pointed Gromov–Hausdorff convergence, see [22, §2.2] or [8, Chapter 7].

Definition 2.4. Let (X, d, μ) be a proper metric measure space. A pointed proper metric measure space (Y, y, ρ, ν) is called a *weak tangent* of (X, d, μ) if there exists a sequence of points $\{x_n\} \subset X$ and constants $\delta_n > 0, \lambda_n > 0$, so that the pointed proper metric measure spaces $\{(X, x_n, \frac{1}{\delta_n}d, \frac{1}{\lambda_n}\mu_n)\}$ converge to (Y, y, ρ, ν) .

We do not require that $\delta_n \rightarrow 0$. In the event that this occurs, we call the limit space a *strict weak tangent* of (X, d, μ) . If $x_n = x \in X$ for all n , we call (Y, y, ρ, ν) a *tangent* to X at x . The notion of *strict tangent* is defined similarly.

Poincaré inequalities persist under Gromov–Hausdorff convergence; see Cheeger [10, §9]. We state here a version of this result due to Keith [22], in a form which is suitable for our setting. For another version, see Koskela [26].

Theorem 2.5 (Cheeger, Koskela, Keith). *Suppose $X_1 \supset X_2 \supset \dots$ are subsets of \mathbb{R}^2 , and for each $n \in \mathbb{N}$, μ_n is a doubling measure supported on X_n , with uniform doubling constant. Let $X = \bigcap_{n \in \mathbb{N}} X_n$, and suppose that the measures $\{\mu_n\}$ converge weakly to a measure μ supported on X . If each (X_n, d, μ_n) supports a p -Poincaré inequality with uniform constants, then (X, d, μ) also supports a p -Poincaré inequality.*

3. DEFINITION AND BASIC PROPERTIES OF THE CARPETS $S_{\mathbf{a}}$

We review the construction of the carpets $S_{\mathbf{a}}$. Fix a sequence

$$\mathbf{a} = (a_1, a_2, \dots)$$

where each a_m is an element of the set $\{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\}$. Starting from the unit square $T_0 = [0, 1]^2$ we set the *level parameter* $m = 1$ and iteratively apply the following two steps:

- Divide each current square into a_m^{-2} essentially disjoint closed congruent subsquares, where m denotes the current level parameter, and remove the central (concentric) subsquare from each square,
- Increase the level parameter m by 1.

We let \mathcal{T}_m denote the collection of all remaining level m squares. For each m , \mathcal{T}_m consists of

$$\prod_{j=1}^m (a_j^{-2} - 1)$$

essentially disjoint closed squares, each of side length

$$s_m := \prod_{j=1}^m a_j.$$

The union of all squares in \mathcal{T}_m is the *level m precarpet*, denoted $S_{\mathbf{a},m}$. A *peripheral square* is a connected component of the boundary of a precarpet. Finally,

$$S_{\mathbf{a}} := \bigcap_{m \geq 0} S_{\mathbf{a},m} = \bigcap_{m \geq 0} \bigcup \mathcal{T}_m.$$

Each carpet $S_{\mathbf{a}}$ is quasiconvex; this can be demonstrated using curves built by countable concatenations of horizontal and vertical segments. It is well-known that the usual Sierpiński carpet $S_{1/3}$ contains other nontrivial line segments, neither horizontal or vertical. Indeed, $S_{1/3}$ contains nontrivial line segments of each of the following slopes: 0, 1/2, 1, 2 and ∞ . For an explicit description of the set of slopes of nontrivial line segments in all carpets $S_{\mathbf{a}}$ in terms of Farey fractions, see [13].

3.1. The natural measure on $S_{\mathbf{a}}$. There is a natural probability measure on $S_{\mathbf{a}}$. Since each precarpet $S_{\mathbf{a},m}$ has positive area, we define a measure μ_m on $[0, 1]^2$ which is the Lebesgue measure restricted to the set $S_{\mathbf{a},m}$, renormalized to have total measure one. The sequence of measures (μ_m) converges weakly to a probability measure μ with support $S_{\mathbf{a}}$. To see this, note that on each (closed) square T of scale s_m that is not discarded, we have $\mu_n(T) = \mu_m(T)$ for all $n \geq m$, since later renormalizations merely redistribute mass within T . Therefore,

$$\mu(T) = \mu_m(T) = \prod_{j=1}^m (a_j^{-2} - 1)^{-1} =: v_m.$$

Moreover, for fixed $Q > 0$,

$$\frac{\mu(T)}{s_m^Q} = \prod_{j=1}^m a_j^{2-Q} (1 - a_j^2)^{-1}.$$

Note that if all $a_m = 1/(2k+1)$, then $\mu(T) = s_m^{Q_k}$ for all $T \in \mathcal{T}_m$ and $\dim S_{1/(2k+1)} = Q_k$. Here Q_k denotes the value in (1.2).

The following proposition describes the basic properties of μ . We write $a \lesssim b$ to mean that there exists a constant $C > 0$ so that $a \leq Cb$, where C depends only on the relevant data. Also, the notation $a \asymp b$ signifies that $a \lesssim b$ and $b \lesssim a$.

Proposition 3.1. *The metric measure space $(S_{\mathbf{a}}, d, \mu)$ has the following properties:*

- (i) *For any \mathbf{a} , μ is a doubling measure.*
- (ii) *For any \mathbf{a} , we have the lower mass bound $\mu(B(x, r)) \gtrsim r^2$ for all x and $r \leq 1$.*
- (iii) *If $\mathbf{a} \in c_0$, then for any $Q < 2$ we have $\mu(B(x, r)) \lesssim r^Q$ for all x and $r > 0$, hence $\dim S_{\mathbf{a}} = 2$.*
- (iv) *If $\mathbf{a} \in \ell^2$, then μ is comparable to Lebesgue measure with constant depending only on $\|\mathbf{a}\|_2$. Moreover, in this case, μ is an Ahlfors 2-regular measure on $S_{\mathbf{a}}$.*
- (v) *If $\mathbf{a} = (a_m)$ is eventually constant (and equal to $\frac{1}{2k+1}$), then μ is comparable to the Hausdorff measure \mathcal{H}^{Q_k} and is an Ahlfors Q_k -regular measure on $S_{\mathbf{a}}$.*

For $x \in S_{\mathbf{a}}$ and $r > 0$ define two integers $m(x, r)$ and $m(r)$ as follows:

- (1) $m(x, r)$ is the smallest integer m so that there exists $T \in \mathcal{T}_m$ with $x \in T \subset B(x, r)$,
- (2) $m(r)$ is the smallest integer m so that $s_m \leq r$.

First, an easy lemma:

Lemma 3.2. *For any x and r , $m(\sqrt{2}r) \leq m(x, r) \leq m(\frac{r}{\sqrt{2}}) + 1$.*

Proof. If $T \in \mathcal{T}_{m(x,r)}$ satisfies $x \in T \subset B(x, r)$, then $\sqrt{2}s_{m(x,r)} = \text{diam } T \leq \text{diam } B(x, r) \leq 2r$ which implies that $s_{m(x,r)} \leq \sqrt{2}r$ and $m(\sqrt{2}r) \leq m(x, r)$. Since $x \in T$ for some $T \in \mathcal{T}_{m(r/\sqrt{2})+1}$ and $\text{diam } T \leq \frac{r}{\sqrt{2}}$, we have $m(x, r) \leq m(\frac{r}{\sqrt{2}}) + 1$. \square

We will derive the various parts of Proposition 3.1 from the following

Proposition 3.3. *For each $x \in S_{\mathbf{a}}$ and $0 < r \leq 1$,*

$$\mu(B(x, r)) \asymp h(r) := r^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right).$$

Proof of Proposition 3.1. Note that $m(r)$ is a decreasing function of r . Part (i) follows easily:

$$\mu(B(x, 2r)) \lesssim (2r)^2 \prod_{j=1}^{m(2r)} \left(\frac{1}{1 - a_j^2} \right) \leq 4r^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \lesssim \mu(B(x, r)).$$

Part (ii) is also clear, since the finite product term in the definition of $h(r)$ is always greater than or equal to one.

Next, we assert that $m(r) \leq m(2r) + 1$ for all $r > 0$. If not, we have $m(r) \geq m(2r) + 2$, so $m(r) - 1 \geq m(2r) + 1$, thus

$$r < s_{m(r)-1} \leq s_{m(2r)+1} \leq \frac{1}{3} s_{m(2r)} \leq \frac{1}{3} \cdot 2r,$$

a contradiction.

We now turn to part (iii). Assume that $\mathbf{a} \in c_0$, i.e., $a_m \rightarrow 0$. We will show that $\limsup_{r \rightarrow 0} \frac{\mu(B(x, r))}{r^Q}$ is finite for each $Q < 2$, uniformly in x . It suffices to show that

$$\limsup_{r \rightarrow 0} \frac{h(r)}{r^Q} < \infty.$$

First we verify that $m(r) \leq -\log_2(r) + 1$. Suppose that n is the largest integer so that $2^n r \leq 1$. Since $m(1) = 0$,

$$m(r) \leq m(2r) + 1 \leq \dots \leq m(2^{n+1}r) + n + 1 \leq m(1) + n + 1 = n + 1 \leq -\log_2(r) + 1.$$

Now, since $a_j \rightarrow 0$, for any $\epsilon > 0$ there exists some $C = C(\epsilon)$ so that

$$\prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \leq C(1 + \epsilon)^{m(r)} \leq C(1 + \epsilon)^{-\log_2 r + 1} \lesssim r^{-\log_2(1 + \epsilon)}.$$

If we choose ϵ so that $2 - Q > \log_2(1 + \epsilon)$, then we are done. From here part (iii) follows easily. Parts (iv) and (v) were discussed in the introduction. \square

Proof of Proposition 3.3. It is straightforward to bound $\mu(B(x, r))$ from above: cover $B(x, r)$ by squares from $\mathcal{T}_{m(r)}$. Then, as $s_{m(r)} \leq r$,

$$\mu(B(x, r)) \leq \frac{(3r)^2}{s_{m(r)}^2} \cdot v_{m(r)} \leq \frac{9r^2}{s_{m(r)}^2} \cdot s_{m(r)}^2 \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \lesssim h(r).$$

To bound $\mu(B(x, r))$ from below, we split the proof into two cases.

Case 1. $r \leq 100s_{m(x, r)}$.

Since $B(x, r)$ contains a square of side $s_{m(x, r)}$, we use the obvious bound $\mu(B(x, r)) \geq v_{m(x, r)}$. Note that $m(r) - 1 \leq m(\frac{r}{\sqrt{2}}) - 1 \leq m(\sqrt{2}r) \leq m(x, r)$. Now,

$$v_{m(x, r)} = s_{m(x, r)}^2 \prod_{j=1}^{m(x, r)} \left(\frac{1}{1 - a_j^2} \right) \geq \left(\frac{1}{100} \right)^2 r^2 \cdot (1 - a_{m(r)}^2) \prod_{j=1}^{m(r)} \left(\frac{1}{1 - a_j^2} \right) \gtrsim h(r).$$

Case 2. $r > 100s_{m(x,r)}$.

Choose $T \in \mathcal{T}_{m(x,r)-1}$ so that $x \in T$. Since $T \not\subseteq B(x, r)$, the side length of T is at least $\frac{r}{\sqrt{2}}$. Since T is a square, $T \cap B(x, r)$ contains a (Euclidean) square V' of side $\frac{r}{4}$. Finally, since $s_{m(x,r)} \leq \frac{r}{100}$ and at most one square of generation $m(x, r)$ is deleted in T , V' contains a square V of side $s_v \in [\frac{r}{32}, \frac{r}{16}]$ consisting entirely of squares from $\mathcal{T}_{m(x,r)}$.

From the preceding facts we conclude that

$$\begin{aligned} \mu(B(x, r)) &\geq \mu(V) = \left(\frac{s_v}{s_{m(x,r)}}\right)^2 v_{m(x,r)} = s_v^2 \cdot \prod_{j=1}^{m(x,r)} \left(\frac{1}{1-a_j^2}\right) \\ &\geq \frac{r^2}{32^2} \cdot (1-a_{m(r)}^2) \prod_{j=1}^{m(r)} \left(\frac{1}{1-a_j^2}\right) \gtrsim h(r). \end{aligned}$$

The proof is finished. \square

We make a final observation regarding the conformal dimension of $S_{\mathbf{a}}$. Recall that a metric space (X, d) is *minimal for conformal dimension* if its Hausdorff dimension is less than or equal to the Hausdorff dimension of any quasisymmetrically equivalent metric space. The self-similar carpets $S_{1/(2k+1)}$ are not minimal for conformal dimension. This result is a consequence of a theorem of Keith and Laakso [24], see also [29] for a brief recapitulation of the proof.

Corollary 3.4. *If $\mathbf{a} \in c_0$, then $S_{\mathbf{a}}$ is minimal for conformal dimension.*

Proof. By Proposition 3.1(iii), $S_{\mathbf{a}}$ has Hausdorff dimension 2. In a similar way, one shows that the Cantor set $C_{\mathbf{a}}$ has Hausdorff dimension 1. Since $S_{\mathbf{a}}$ contains the product of $C_{\mathbf{a}}$ and an interval, which has Hausdorff dimension 2, by [2, Section 5, Remark 1] the space $S_{\mathbf{a}}$ is minimal for conformal dimension. \square

The conformal dimensions of the carpets $S_{\mathbf{a}}$ when $\mathbf{a} \notin c_0$ remain unknown. Determining the conformal dimension of $S_{1/3}$ is a longstanding open problem.

4. FAILURE OF THE POINCARÉ INEQUALITY

In this section we provide conditions under which the p -Poincaré inequality fails to be satisfied on $S_{\mathbf{a}}$ for various choices of p and \mathbf{a} . In doing so we verify the necessity of the summability criteria in Theorems 1.5 and 1.6.

Proposition 4.1. *If $\mathbf{a} \notin \ell^1$, then $S_{\mathbf{a}}$ does not support a 1-Poincaré inequality.*

Proof. For each $m \in \mathbb{N}$, let $T_m \subset S_{\mathbf{a}}$ be the vertical middle strip of width s_m . Define $f_m : S_{\mathbf{a}} \rightarrow [0, 1]$ to be the function which is 0 to the left of T_m , 1 to the right of T_m and extend it linearly across T_m . This function has upper gradient $\rho_m : S_{\mathbf{a}} \rightarrow [0, \infty]$ which is identically $1/s_m$ on T_m and $\rho_m \equiv 0$ elsewhere. We compute

$$\int \rho_m d\mu = \prod_{i=1}^m \left(a_i^{-1} \cdot \frac{a_i - a_i^2}{1 - a_i^2} \right) = \prod_1^m \frac{1}{1 + a_i}.$$

Since $\mathbf{a} \notin \ell^1$, the right hand side goes to zero as $m \rightarrow \infty$. Observe that $\int_{S_{\mathbf{a}}} f_m d\mu = 1/2$, and f_m takes values of 0 and 1 on a set of measure bounded away from zero independently of m . Therefore, (1.1) cannot be satisfied for $p = 1$ and fixed constant C . \square

We now consider p -Poincaré inequalities with $p > 1$. If $\mathbf{a} \notin \ell^3$, a careful adaptation of the proof of the previous proposition shows that $S_{\mathbf{a}}$ does not support any Poincaré inequality. However, the carpets considered in this paper have a very specific geometry that leads to the following sharp result.

Proposition 4.2. *If $\mathbf{a} \notin \ell^2$, then $S_{\mathbf{a}}$ does not support a p -Poincaré inequality for any $p \geq 1$.*

Proof. Our goal is to build a set $X \subset S_{\mathbf{a}}$ with $\mu(X) = 0$ so that for every rectifiable curve γ joining the left and right hand edges of $S_{\mathbf{a}}$ we have $\int_{\gamma} \rho ds \geq 1$, where ρ is the characteristic function of X .

This suffices to show the failure of the Poincaré inequality, for we can then define a function f on $S_{\mathbf{a}}$ by letting $f(x)$ be the infimum of $\int_{\gamma} \rho ds$, where γ ranges over all rectifiable curves joining the left edge of $S_{\mathbf{a}}$ to x . As $\rho \leq 1$ and $S_{\mathbf{a}}$ is quasi-convex, f is a Lipschitz function which is zero on the left edge of $S_{\mathbf{a}}$. The property described above shows that $f \geq 1$ on the right edge of $S_{\mathbf{a}}$. Since f has an upper gradient with essential supremum zero, we have a contradiction to (1.1).

In the remainder of the proof we build the set X and show it has the desired properties for some fixed, arbitrary rectifiable curve γ in $S_{\mathbf{a}}$ that joins the left and right hand edges of $S_{\mathbf{a}}$. By passing to a subcurve if necessary we may assume that γ is an arc, i.e., that it is injective.

As part of our proof we shall build cut sets which disconnect $S_{\mathbf{a}}$. To simplify our discussion later, we define the set $H_{\mathbf{a}}$ to be the union of sets $[a, b] \times (c, d)$, for every deleted open square $(a, b) \times (c, d)$ in the construction of $S_{\mathbf{a}}$.

Initial step. Let $A_0 = S_{\mathbf{a}}$, and $\Gamma_0 = \{\gamma\}$.

We divide $S_{\mathbf{a}}$ into $m_1 = s_1^{-1}$ vertical strips of width a_1 . These strips are bounded by vertical cut sets V_0, V_1, \dots, V_{m_1} , where $V_0 = (\{0\} \times [0, 1]) \setminus H_{\mathbf{a}}$, $V_1 = (\{a_1^{-1}\} \times [0, 1]) \setminus H_{\mathbf{a}}$, and so on. In other words, each V_j is a vertical line, with the exception that the interiors of vertical sides of the deleted square of side a_1 are not contained in the appropriate V_j .

We now split $\Gamma_0 = \{\gamma\}$ into a disjoint family of curves. We parametrize γ by the interval $[0, 1]$, with $\gamma(0) \in V_0$, and $\gamma(1) \in V_{m_1}$. Let $t_0^+ \geq 0$ be the last time γ meets V_0 . Let $t_1^- > t_0^+$ be the next time after that γ meets V_1 . Let γ_1 be the subpath of γ given by restricting to $[t_0^+, t_1^-]$.

Continue inductively, letting $t_{j-1}^+ \geq t_{j-1}^-$ be the last time γ meets V_{j-1} , and $t_j^- > t_{j-1}^+$ be the next time γ meets V_j . Let γ_j be the subpath of γ given by $[t_{j-1}^+, t_j^-]$.

By construction, $\Gamma_1 = \{\gamma_1, \dots, \gamma_{m_1}\}$ is a family of curves, where each γ_j joins V_{j-1} to V_j , and is contained between them. (See Figure 3, where the deleted subpaths are indicated by dotted lines.)

Note that the length of Γ_1 (i.e., the sum of the lengths of $\gamma_1, \dots, \gamma_{m_1}$), is at least one and at most the length of Γ_0 , that is $\text{length}(\gamma)$.

Inductive step (fold in). Fix $i \geq 1$. We are given as input a collection $\Gamma_i = \{\gamma_j\}$ of $m_i = s_i^{-1}$ curves and vertical slices V_0, \dots, V_{m_i} , where γ_j joins V_{j-1} to V_j and is contained between them, for each $j = 1, \dots, m_i$.

Choose the largest $l_i \in \mathbb{N}$ so that $l_i a_{i+1} \leq \frac{1}{3}$. Note that since $a_{i+1} \in \{\frac{1}{3}, \frac{1}{5}, \dots\}$, we have $l_i a_{i+1} > \frac{1}{3} - \frac{1}{5}$.

Let \mathcal{D}_i be the collection of open rectangles of width $l_i a_{i+1} s_i$ and height $(1 - 2l_i a_{i+1}) s_i$ centered on and adjacent to either the left or right sides of the deleted squares of side length s_i . Consider the squares of side length $l_i a_{i+1} s_i$ above and below each of these rectangles. Each such square S has a diagonal which meets the corner of a deleted square of side length s_i . This diagonal divides S into two triangles; let \mathcal{R}_i be the collection of closures of those triangles that share a side with a deleted square of side s_i . See Figure 4 for part of an example where these regions are labeled. We use the convention that the rectangles and triangles referred to are actually the intersection of the corresponding planar set and the carpet $S_{\mathbf{a}}$.

We now define a folding map $F_i : S_{\mathbf{a}} \rightarrow S_{\mathbf{a}}$ by declaring F_i to be the identity except on $\cup \mathcal{R}_i$, where the map folds each triangle $R \in \mathcal{R}_i$ across the diagonal. Note that the horizontal edges of every rectangle in \mathcal{D}_i are mapped to vertical edges, and that F_i is discontinuous along such edges.

Notice that in Figure 4, if $a_{i+1} < \frac{1}{3}$ then the region $\cup \mathcal{D}_i$ will not overlap with $\cup \mathcal{D}_{i+1}$. However, when $a_{i+1} = \frac{1}{3}$, \mathcal{D}_i may contain a square Q of side length s_{i+1} adjacent to the left or right of a particular deleted square of side length s_{i+1} , but it cannot contain both such squares. When this happens, we leave the square Q untouched at step $i + 1$, and do not include any part of it in \mathcal{D}_{i+1} .

We apply F_i to the collection of curves Γ_i . Consider the resulting collection of curves. (Note that some curves have been broken into smaller pieces along the discontinuities of F_i .) We now build Γ_{i+1} using the same inductive construction as we used in the initial step to build Γ_1 from Γ_0 . As before, let $V_j = (\{j s_{i+1}\} \times [0, 1]) \setminus H_{\mathbf{a}}$, for $j = 0, \dots, m_{i+1}$, separate the unit square into essentially disjoint vertical strips of width s_{i+1} . Here $m_{i+1} := s_{i+1}^{-1}$. We define times

$$0 \leq t_0^+ < t_1^- \leq t_1^+ < \dots$$

as before, for the broken curve $F_i(\Gamma_i)$.

Let γ_j be the broken curve in $F_i(\Gamma_i)$ given by the restriction to $[t_{j-1}^+, t_j^-]$. In fact, γ_j is connected.

By construction, $\Gamma_{i+1} = \{\gamma_1, \dots, \gamma_{m_{i+1}}\}$ is a family of curves, where each γ_j joins V_{j-1} to V_j , and is contained between them.

Moreover, Γ_{i+1} will lie inside $A_i = \overline{A_{i-1} \setminus \cup(\mathcal{D}_i \cup \mathcal{R}_i)}$. To see why this is so, consider (for example) j so that V_j lies on the left edge of a rectangle D in \mathcal{D}_i , where D lies on the left of an omitted square of side length s_i . By definition, t_j^+ is the last time the broken curve $F_i(\Gamma_i)$ meets V_j . This corresponds to the last time that Γ_i meets either V_j or the horizontal edges of any element of \mathcal{D}_i above or below D . Consequently, the family Γ_{i+1} is disjoint from $\cup \mathcal{D}_i$ and (by the definition

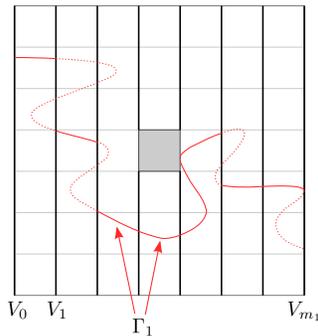


FIGURE 3. Curve splitting

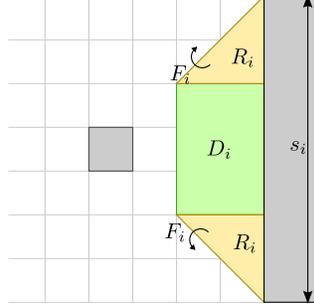


FIGURE 4. Unfolding

of F_i) it is also disjoint from the interior of $\cup \mathcal{R}_i$. As before, the length of Γ_{i+1} is at least one and at most the length of Γ_i .

Conclusion. We continue this construction until $i = n$, when we have a collection of curves Γ_{n+1} that lies in A_n , and have deleted the rectangles in the collections $\mathcal{D}_1, \dots, \mathcal{D}_n$ from the sides of the deleted squares. We let $X_n^{(n)} = A_n$, and now proceed to unfold this set and the curves Γ_{n+1} back out into the regions $\cup \mathcal{R}_1, \dots, \cup \mathcal{R}_n$.

Define inductively

$$X_n^{(i-1)} = F_i^{-1}(X_n^{(i)}),$$

for $i = n, n-1, \dots, 1$. Observe that $X_n^{(0)}$ is all of $S_{\mathbf{a}}$, but with certain rectangles removed that are adjacent to sides of removed squares of $S_{\mathbf{a},n}$.

It is clear that $S_{\mathbf{a}} = X_0^{(0)} \supset X_1^{(0)} \supset X_2^{(0)} \supset \dots$. Let $X = \bigcap_{i=0}^{\infty} X_i^{(0)}$. For each n , and for any γ in Γ , our construction implies that

$$\begin{aligned} \text{length}(\gamma \cap X_n^{(0)}) &= \text{length}(\Gamma_0 \cap X_n^{(0)}) \geq \text{length}(\Gamma_1 \cap X_n^{(0)}) \\ &\geq \text{length}(\Gamma_2 \cap X_n^{(1)}) \geq \dots \geq \text{length}(\Gamma_{n+1} \cap X_n^{(n)}) \\ &= \text{length}(\Gamma_{n+1} \cap A_n) \geq 1, \end{aligned}$$

since Γ_{n+1} is a chain of paths crossing each vertical strip of width s_{n+1} from the left to the right, and Γ_{n+1} lives in A_n .

Therefore, $\text{length}(\gamma \cap X) \geq 1$, since $\text{length} = \mathcal{H}^1$ is a measure on arcs.

Let ρ be the characteristic function of X . Since γ was an arbitrary rectifiable curve joining the left and right sides of $S_{\mathbf{a}}$, and X was constructed independently of γ , we have shown that $\int_{\gamma} \rho ds \geq 1$ for every such curve γ .

It remains to prove that $\mu(X) = 0$. Consider each deleted square of side s_i . Out of the neighboring $(a_i^{-2} - 1)$ boxes of side s_i , from at least one (two if $a_i \neq \frac{1}{3}$) of these we will delete a rectangle in \mathcal{D}_i whose μ -measure, as a proportion of a square of side s_i , is at least $(1 - 2l_i a_{i+1})l_i a_{i+1} \geq \frac{2}{45}$. Since all the rectangles in \mathcal{D}_i are pairwise disjoint, we have

$$\mu(X) \leq \prod_{i=1}^{\infty} \left(1 - \frac{1}{a_i^{-2} - 1} \cdot \frac{2}{45} \right) = \prod_{i=1}^{\infty} \left(1 - \frac{2}{45} a_i^2 + \dots \right),$$

which converges to zero since $\mathbf{a} \notin \ell^2$. This completes the proof. \square

Remark 4.3. The argument also shows that $S_{\mathbf{a}}$ does not support an ∞ -Poincaré inequality. See [12] for the definition, which is weaker than the p -Poincaré inequality for any finite p .

4.1. Weak tangents of Sierpiński carpets. Weak tangents of metric spaces describe infinitesimal behavior at a point or along a sequence of points. In this section we characterize the strict weak tangents of non-self-similar carpets. More precisely, we prove the following proposition.

Proposition 4.4. *Let $\mathbf{a} \in c_0$. Then every strict weak tangent of $S_{\mathbf{a}}$ is of the form $(\mathbb{R}^2 \setminus T, d, \nu)$ where T is a generalized square and ν is proportional to Lebesgue measure restricted to $\mathbb{R}^2 \setminus T$.*

By a *generalized square* we mean a set $T \subsetneq \mathbb{R}^2$ of the type

$$T = (a, b) \times (c, d)$$

where $-\infty \leq a \leq b \leq \infty$, $-\infty \leq c \leq d \leq \infty$, and $b - a = d - c$ if one (hence both) of these values is finite. (We interpret the degenerate interval (a, b) , $a = b$, as the empty set.) Thus T is either the empty set, an open square, a quadrant or a half-space.

Suppose W is a strict weak tangent arising as the limit of the sequence of metric spaces $\{X_n = (S_{\mathbf{a}}, x_n, \frac{1}{\delta_n}d)\}$, where $x_n \in S_{\mathbf{a}}$, $\delta_n \in (0, \infty)$, and $\delta_n \rightarrow 0$.

The following lemma indicates why W can omit at most one large square.

Lemma 4.5. *There exist $R_n \rightarrow \infty$ and $r_n \rightarrow 0$ so that in the ball $B(x_n, R_n) \subset X_n$ there is at most one square of side greater than 1 removed, and all other squares removed have size at most r_n .*

Proof. Fix n , and let $m = m(\delta_n)$, i.e., $s_m \leq \delta_n < s_{m-1}$.

Either $\delta_n \in [s_m, s_{m-1}\sqrt{a_m})$, or $\delta_n \in [s_{m-1}\sqrt{a_m}, s_{m-1})$. In the first case, removed squares of size at least $\frac{s_m}{\delta_n}$ are $\frac{s_{m-1}}{2\delta_n} \geq \frac{1}{2\sqrt{a_m}}$ separated, while all others have size at most $\frac{s_{m+1}}{\delta_n} \leq a_{m+1}$. In the second case, removed squares of size at least $\frac{s_{m-1}}{\delta_n} \geq 1$ are $\frac{s_{m-2}}{2\delta_n} \geq \frac{1}{2a_{m-1}}$ separated, and all others have size at most $\frac{s_m}{\delta_n} \leq \sqrt{a_m}$.

Setting $R_n = \frac{1}{4} \min\{\frac{1}{a_{m-1}}, \frac{1}{\sqrt{a_m}}\}$ and $r_n = \max\{\sqrt{a_m}, a_{m+1}\}$, we have proved the lemma. \square

Proof of Proposition 4.4. Using the preceding lemma, we can reduce the proof of Proposition 4.4 to consideration of limits of $\mathbb{R}^2 \setminus T_n$ where T_n is either a square of side at least one, or the empty set. It is easy to see that any strict weak tangent W as above will be isometric to $\mathbb{R}^2 \setminus T$, where T is either a square (of side at least one), a quarter-plane, half-plane or the empty set.

Since the measure μ on $S_{\mathbf{a}}$ agrees with the weak limit of renormalized Lebesgue measure on the domains $S_{\mathbf{a},m}$, by the lemma, if we look at measures of balls in X_n of size much larger than r_n , they will agree with a constant multiple of Lebesgue measure up to small error. Consequently, the possible measures on W will arise as limits of rescaled Lebesgue measure on $\mathbb{R}^2 \setminus T_n$. The only possible non-trivial Radon measure of this type is a constant multiple of Lebesgue measure restricted to $\mathbb{R}^2 \setminus T$. This finishes the proof of Proposition 4.4. \square

Note that all of the weak tangent spaces identified in the conclusion of Proposition 4.4 support a 1-Poincaré inequality, with uniform constants (i.e., independent of the choice of such a weak tangent space). This is because we have only a finite number of similarity types of spaces (full space, half space, quarter space, or the complement of a square), and the Poincaré inequality data is invariant under similarities. The quasiconvexity of the original carpets $S_{\mathbf{a}}$ is a standard

fact. Indeed, arbitrary pairs of points can be joined by quasiconvex curves which are comprised of countable unions of horizontal and vertical line segments.

Definition 4.6. A metric space (X, d) is *locally Gromov–Hausdorff close to planar domains* if for each $x \in X$ and each $\epsilon > 0$, there exists $r > 0$ and a domain $\Omega \subset \mathbb{R}^2$ so that the Gromov–Hausdorff distance between the metric ball $B(x, r) \subset X$ and Ω is at most ϵ . Furthermore, (X, d) is *uniformly locally Gromov–Hausdorff close to planar domains* if ϵ can be chosen independently of x .

The fact that $S_{\mathbf{a}}$ is uniformly locally Gromov–Hausdorff close to planar domains follows easily from the construction and the condition $\mathbf{a} \in c_0$.

The preceding discussion and Theorem 1.6 understood, the proof of Corollary 1.8 is complete by the choice of $\mathbf{a} \in c_0 \setminus \ell^2$.

5. VALIDITY OF THE POINCARÉ INEQUALITY: THE CASE $\mathbf{a} \in \ell^1$

In this section, we make the standing assumption that $\mathbf{a} \in \ell^1$, and show that $S_{\mathbf{a}}$, equipped with the Euclidean metric and the canonical measure described in subsection 3.1, admits a 1-Poincaré inequality. Recall that whenever \mathbf{a} is in $\ell^2 \subset \ell^1$, Proposition 3.1(iv) states that μ is comparable to Hausdorff 2-measure \mathcal{H}^2 restricted to $S_{\mathbf{a}}$. For simplicity we will work with \mathcal{H}^2 in this section and the next.

According to Theorem 2.1, the validity of a Poincaré inequality is equivalent to the existence of curve families of uniformly and quantitatively large weighted modulus joining arbitrary pairs of points. The desired curve family must spread out as it escapes from the endpoints. This divergence is measured via transversal measures on the edges of squares in the precarpets.

We explicitly construct this family using the structure of the carpet $S_{\mathbf{a}}$. In subsection 5.1 we state and prove four lemmas providing the ‘building blocks’ of the construction. Each of these building blocks consists of families of disjoint curves joining edges of certain squares in the carpet. These families are each equipped with a natural transversal measure, and concatenated to produce the desired family connecting the given endpoints.

By Theorem 2.5, to demonstrate that $S_{\mathbf{a}}$ admits a 1-Poincaré inequality, it suffices to prove that the precarpets $\{(S_{\mathbf{a},m}, d, \mu_m)\}_{m \in \mathbb{N}}$ support a 1-Poincaré inequality with constants independent of m . In order to simplify the discussion, we work in a fixed precarpet $S_{\mathbf{a},M}$. In subsection 5.2 we use the ‘building block’ lemmas of subsection 5.1 to build the desired path family in $S_{\mathbf{a},M}$, and complete the proof of our main theorem.

To simplify the argument we will impose the requirement

$$(5.1) \quad a_i \leq a_* := \frac{1}{20} \quad \text{for all } i.$$

This requirement entails no loss of generality, as the carpet $S_{\mathbf{a}}$ is the finite union of similar copies of some carpet $S_{\mathbf{a}'}$, where $\mathbf{a}' \in \ell^1$ and all entries of \mathbf{a}' are less than a_* , glued along their boundaries. By Theorem 2.2, if each of these smaller carpets supports a 1-Poincaré inequality, then the original carpet will also. We note that the gluing procedure in Theorem 2.2 differs from the union considered here, however, the resulting metrics are bi-Lipschitz equivalent and the validity of Poincaré inequalities is unaltered by this change of metric. The constants for the overall Poincaré inequality will depend on the number of copies which are glued together, which in turn depends on how far out in the sequence \mathbf{a} we must go to ensure condition (5.1). If \mathbf{a} is monotone decreasing, this data depends only on $\|\mathbf{a}\|_1$.

If (5.1) is not satisfied, the algorithmic construction in the proof of Lemma 5.9 becomes slightly more complicated, however, the rest of the argument is unchanged. We leave such modifications to the industrious reader.

5.1. Building block lemmas. Recall that \mathcal{T}_m denotes the collection of all level m squares in the construction of the carpet $S_{\mathbf{a}}$.

Definition 5.1. Fix $m \in \mathbb{N}$ and a square $T' \in \mathcal{T}_{m-1}$. For non-negative integers a, b, k and l , a set $\mathcal{C} = [as_m, (a+k)s_m] \times [bs_m, (b+l)s_m]$ is called a k by l block in T' if it is contained in T' and does not contain the removed central subsquare of T' . We will often choose a preferred edge L of a block \mathcal{C} that does not contain the boundary of the removed central subsquare of T' , and declare it to be the *leading edge* of \mathcal{C} . The pair (\mathcal{C}, L) is called a *directed block in T'* . A directed 1 by 1 block in T' is called a *directed square in T'* . We will suppress reference to T' if the dependence is clear or unimportant.

Note that the choice of a leading edge of a block gives rise to an outward-pointing unit normal vector $(1, 0)$, $(0, 1)$, $(-1, 0)$ or $(0, -1)$.

Directed blocks (possibly in different squares or even of different generations) are *coherent* if the corresponding outward-pointing unit normal vectors coincide.

We say that the directed block (\mathcal{C}_2, L_2) follows the directed block (\mathcal{C}_1, L_1) if $\mathcal{C}_1 \cap \mathcal{C}_2 = L_1$ and $L_1 \not\subseteq L_2$.

We introduce a distinguished set π_M which will parameterize certain curve families. Let π_M be the set of all $x \in [0, 1]$ with the property that the line $\{x\} \times \mathbb{R}$ does not meet the interior or left hand side of a peripheral square removed in the construction of $S_{\mathbf{a}, M}$. Let (\mathcal{C}, L) be a directed block in a square $T' \in \mathcal{T}_{m-1}$. There is a unique orientation preserving isometry $i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $(0, 0) \in i(\mathcal{C}) \subset [0, 1]^2$, and so that $i(L)$ is contained in the x -axis. We define $\pi_M(L)$ to be the union of $L \cap i^{-1}(\pi_M)$ with the endpoints of L . It follows from the assumption $\mathbf{a} \in \ell^1$ that

$$(5.2) \quad \mathcal{H}^1(\pi_M(L)) \asymp \text{diam}(L).$$

Given two such sets $\pi_M(L_1)$ and $\pi_M(L_2)$ arising from isometries i_1 and i_2 , there is a unique bijection $h : \pi_M(L_1) \rightarrow \pi_M(L_2)$ so that $i_2 \circ h \circ i_1^{-1}$ is an order-preserving, piecewise linear bijection from $i_1(\pi_M(L_1)) \subset \mathbb{R}$ to $i_2(\pi_M(L_2)) \subset \mathbb{R}$ with a.e. constant derivative. We call h the natural *ordered bijection*.

Definition 5.2. Let E be a Borel subset of a side of a block \mathcal{C} such that $0 < \mathcal{H}^1(E) < \infty$. A *path family on E (in \mathcal{C})* is a collection of disjoint curves $\Gamma = \{\gamma_z\}_{z \in E}$ in $\mathcal{C} \cap S_{\mathbf{a}, M}$ with the property that $\gamma_z(0) = z$ for all $z \in E$. We also require that the measure

$$\nu_\Gamma(A) := \frac{1}{\mathcal{H}^1(E)} \int_E \mathcal{H}^1(A \cap \gamma_z) d\mathcal{H}^1(z),$$

is a Borel measure on \mathbb{R}^2 .

As previously discussed, we will construct curve families of uniformly and quantitatively large weighted modulus joining arbitrary pairs of points in $S_{\mathbf{a}}$. The following notion of ∞ -connection quantifies the degree to which these curve families must spread out as they escape from the endpoints, measured with respect to the L^∞ norm. The L^∞ norm arises here by Hölder duality, as we are proving the 1-Poincaré inequality. In the following section, we will introduce the analogous notion of q -connection for finite q in order to address the case of the p -Poincaré inequality for $p > 1$.

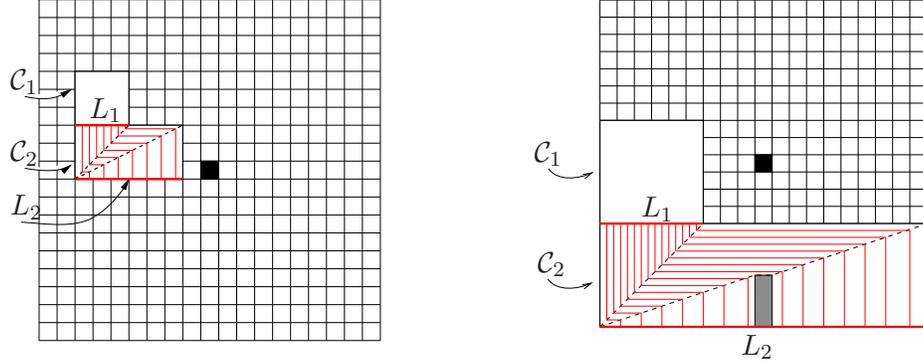


FIGURE 5. (a) Expanding to a larger interval; (b) Expanding to the parent generation

Definition 5.3. Suppose that the directed block (\mathcal{C}_2, L_2) follows the directed block (\mathcal{C}_1, L_1) , and let $h: \pi_M(L_1) \rightarrow \pi_M(L_2)$ be the natural ordered bijection. A path family Γ on $\pi_M(L_1)$ in \mathcal{C}_2 is called an ∞ -connection from L_1 to L_2 through \mathcal{C}_2 (with constant C) if $\gamma(1) = h(\gamma(0))$ for each $\gamma \in \Gamma$, and if $\nu_\Gamma \ll \mathcal{H}^2 \llcorner \text{spt}(\Gamma)$ with

$$(5.3) \quad \left\| \frac{d\nu_\Gamma}{d\mathcal{H}^2} \right\|_{L^\infty(\mathcal{C}_2; \mathcal{H}^2)} \leq \frac{C}{\mathcal{H}^1(\pi_M(L_1))}.$$

For the remainder of this subsection, we fix $0 < m \leq M$, and directed blocks (\mathcal{C}_2, L_2) following (\mathcal{C}_1, L_1) in a directed square $(T', L') \in \mathcal{T}_{m-1}$. We declare the *central column* of T' to be the central row or column of T' that intersects L' .

We now state our building block lemmas; see figures 5 and 6.

Lemma 5.4 (Expanding). *Suppose that*

- (\mathcal{C}_1, L_1) and (\mathcal{C}_2, L_2) are coherent with (T', L') ,
- the sides of \mathcal{C}_2 perpendicular to L_2 have length equal to that of L_1 , and
- it holds that $\mathcal{H}^1(L_2)/\mathcal{H}^1(L_1) \leq 10$.

Then there is an ∞ -connection Γ from L_1 to L_2 through \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_1)$.

Lemma 5.5 (Expanding to the parent generation). *Suppose that*

- (\mathcal{C}_1, L_1) is coherent with (T', L_2) , where $L_2 = L'$,
- the length of L_1 is equal to the length of an edge of \mathcal{C}_2 perpendicular to L_2 ,
- L_1 intersects an edge of T' perpendicular to L' , and
- it holds that $\mathcal{H}^1(L_2)/\mathcal{H}^1(L_1) \leq 10$.

Then there is an ∞ -connection Γ from L_1 to L_2 through \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_1)$.

Lemma 5.6 (Turning). *Suppose that*

- L_1 and L_2 are perpendicular, and
- all edges of \mathcal{C}_2 have length equal to the length of L_1 .

Then there is an ∞ -connection Γ from L_1 to L_2 through \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_1)$.

Lemma 5.7 (Going straight). *Suppose that*

- (\mathcal{C}_1, L_1) and (\mathcal{C}_2, L_2) are coherent with (T', L') ,

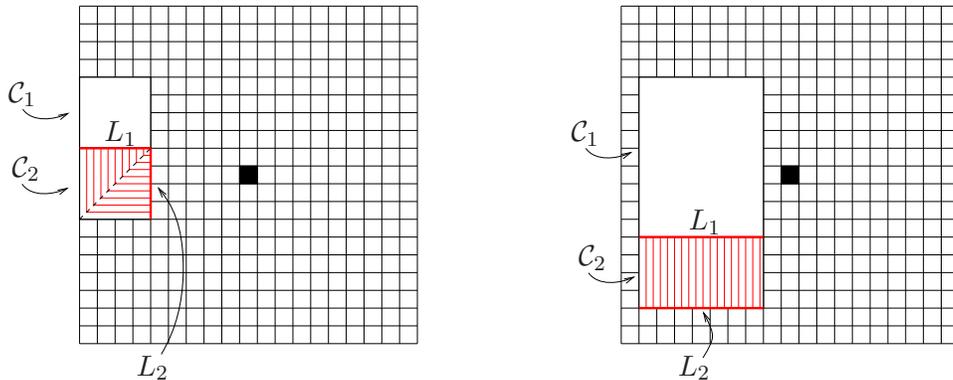


FIGURE 6. (a) Turning; (b) Going straight

- L_1 and L_2 are of equal length, and
- the sides of C_2 perpendicular to L_2 have length in $[\mathcal{H}^1(L_1)/2, 10\mathcal{H}^1(L_1)]$.

Then there is an ∞ -connection Γ from L_1 to L_2 through C_2 with constant $C = C(\|\mathbf{a}\|_1)$.

Proof of Lemma 5.4. We assume that $T' = [0, s_{m-1}]^2$ and $L' = [0, s_{m-1}] \times \{0\}$. We further assume that $C_2 = [0, as_m] \times [0, bs_m]$ and $C_1 = [0, bs_m] \times [bs_m, cs_m]$, as well as that $L_2 = [0, as_m] \times \{0\}$ and $L_1 = [0, as_m] \times \{bs_m\}$. Here a , b , and c are positive integers with $a \geq b$ and $c > b$. For ease of notation, set $E = \pi_M(L_1)$ and $F = \pi_M(L_2)$. Note that the natural ordered bijection $h: E \rightarrow F$ satisfies

$$h'(z) = \frac{\mathcal{H}^1(F)}{\mathcal{H}^1(E)} \geq 1$$

for every interior point z of E , i.e., for all but finitely many points. In the case that $m = M$, the function h is affine.

We now define a path family Γ on E . Given $z = (u, bs_m) \in E$, let γ_z^1 be the vertical line segment connecting (u, bs_m) to (u, u) , let γ_z^2 be the horizontal line segment connecting (u, u) to $h(z) + (0, u)$, and let γ_z^3 be the vertical line segment connecting $h(z) + (0, u)$ to $h(z)$. Let γ_z be the concatenation of γ_z^1 , γ_z^2 and γ_z^3 ; then $\gamma_z \subseteq S_{\mathbf{a}, M} \cap C_2$. Let $\Gamma = \{\gamma_z : z \in E\}$.

We may write the support of Γ as the union of the supports of the curve families

$$\Gamma^i = \{\gamma_z^i\}_{z \in E}, \quad i = 1, 2, 3.$$

Given a set A contained in the support of Γ , we write $A^i = A \cap \text{spt } \Gamma^i$.

For $i = 1$ or 2 , Fubini's theorem yields $\mathcal{H}^2(A^i) = \mathcal{H}^1(E)\nu_\Gamma(A^i)$. For $i = 3$, a simple change of variables shows that

$$\mathcal{H}^2(A^3) = \mathcal{H}^1(F)\nu_\Gamma(A^3) \geq \mathcal{H}^1(E)\nu_\Gamma(A^3).$$

Together, this shows that Γ is an ∞ -connection from L_1 to L_2 through C_2 with constant 1. \square

We omit the proofs of Lemmas 5.5–5.7 as they are nearly identical to that of Lemma 5.4.

5.2. Verification of the 1-Poincaré inequality. We are now ready to prove the following proposition.

Proposition 5.8. *Suppose that $\mathbf{a} \in \ell^1$. Then $S_{\mathbf{a}}$ admits a 1-Poincaré inequality.*

Proof. As mentioned in the introduction to this section, it is enough to prove that for fixed M , the precarpet $(S_{\mathbf{a},M}, d, \mu_M)$ supports a 1-Poincaré inequality with constant independent of M . Towards this end, we take advantage of Theorem 2.1. The constant C_2 in Keith's condition (2.1) will be an absolute quantity which could in principle be computed explicitly as a fixed multiple of the implicit multiplicative constant in conditions (5) and (6) of Lemma 5.9 below. On the other hand, the constant C_1 in (2.1) depends on C_0 in Lemma 5.9, which in turn depends on the constants in Lemmas 5.4–5.7 above. In particular, C_1 will depend heavily on $\|\mathbf{a}\|_1$.

In order to verify the condition in Theorem 2.1, let us fix $x, y \in S_{\mathbf{a},M}$ with $x \neq y$. If $|x-y| < 10s_M$, then we are in the Euclidean situation with possibly a square removed nearby, so (2.1) holds with uniform constants. Let us assume that for some $m \leq M$ we have $10s_m \leq |x-y| < 10s_{m-1}$.

The implicit multiplicative constants in conditions (5) and (6) below are fixed, universal quantities which could be explicitly computed; to simplify the story we have spared the reader any explicit calculation. For instance, both of these multiplicative constants can be chosen to be 100.

Lemma 5.9. *There exist integers $K_- < 0 < K_+$, a sequence of directed blocks $\{(\mathcal{C}_i, L_i)\}_{i=K_-}^{K_+}$, and path families Γ_i each supported on \mathcal{C}_i , with the following properties, for some uniform constant C_0 .*

- (1) \mathcal{C}_{K_-} and \mathcal{C}_{K_+} are 2 by 1 blocks (or 1 by 2 blocks) on scale s_M containing x and y respectively.
- (2) $\text{dist}(x, L_{K_-}) \geq s_M/2$ and $\text{dist}(y, L_{K_+}) \geq s_M/2$.
- (3) Γ_{K_-} and Γ_{K_+} consist of the collection of straight lines joining x to L_{K_-} and y to L_{K_+} respectively.
- (4) For each $i = (K_- + 1), \dots, -1$, (\mathcal{C}_i, L_i) follows $(\mathcal{C}_{i-1}, L_{i-1})$; for each $i = 1, \dots, K_+ - 1$, (\mathcal{C}_i, L_i) follows $(\mathcal{C}_{i+1}, L_{i+1})$; (\mathcal{C}_0, L_1) follows $(\mathcal{C}_{-1}, L_{-1})$, and (\mathcal{C}_0, L_{-1}) follows (\mathcal{C}_1, L_1) . Moreover, Γ_i is an ∞ -connection from L_{i-1} to L_i through \mathcal{C}_i (if $i < 0$), from L_{i+1} to L_i through \mathcal{C}_i (if $i > 0$), or from L_{-1} to L_1 through \mathcal{C}_0 (if $i = 0$), all with constant C_0 .
- (5) For each $i = (K_- + 1), \dots, (K_+ - 1)$,

$$\min\{\text{dist}(x, \mathcal{C}_i), \text{dist}(y, \mathcal{C}_i)\} \asymp \text{diam}(\mathcal{C}_i) \asymp \text{diam}(L_i).$$

- (6) $\sum_{i=K_-}^{K_+} \text{diam}(\mathcal{C}_i) \asymp |x-y|$.
- (7) The blocks $\mathcal{C}_{K_-}, \dots, \mathcal{C}_{K_+}$ are essentially disjoint.

Defining L_{start} and L_{finish} appropriately according to each case, we summarize Lemma 5.9(4) by stating that Γ_i is an ∞ -connection from L_{start} to L_{finish} in \mathcal{C}_i for each $i = (K_- + 1), \dots, K_+ - 1$.

We postpone the proof of this lemma.

The path families $\Gamma_{K_-}, \dots, \Gamma_{K_+}$ concatenate together by gluing paths using the natural ordered bijection on each block. This gives a path family Γ consisting of pairwise disjoint, rectifiable curves joining x to y , carrying a probability measure $\sigma = \sigma_\Gamma$ on Γ which agrees with σ_{Γ_i} for each i on \mathcal{C}_i . The measure $\nu = \nu_\Gamma$ on the support of Γ defined by

$$(5.4) \quad \nu(A) = \int_{\Gamma} \mathcal{H}^1(A \cap \gamma) d\sigma$$

restricts to ν_{Γ_i} on each \mathcal{C}_i .

This measure ν is absolutely continuous with respect to μ_{xy} . For $i = K_-, \dots, K_+$, we have the following bound on the Radon–Nikodym derivative $\frac{d\nu}{d\mu_{xy}}$. We have $\text{diam}(\mathcal{C}_i) \asymp \min\{\text{dist}(x, \mathcal{C}_i), \text{dist}(y, \mathcal{C}_i)\}$, and so $\mu_{xy} \asymp \frac{1}{\text{diam}(\mathcal{C}_i)} \mathcal{H}^2$ on \mathcal{C}_i . Therefore

$$\begin{aligned} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^\infty(\mathcal{C}_i; \mu_{xy})} &\asymp \left\| \frac{d\nu}{d\mathcal{H}^2} \right\|_{L^\infty(\mathcal{C}_i; \mathcal{H}^2)} \text{diam}(\mathcal{C}_i) \\ &\lesssim (\mathcal{H}^1(\pi_M(L_i)))^{-1} \text{diam}(\mathcal{C}_i) \lesssim 1. \end{aligned}$$

This bound also holds on \mathcal{C}_{K_-} and \mathcal{C}_{K_+} : note that on $\text{spt}(\Gamma_{K_-}) \subset \mathcal{C}_{K_-}$, both ν and μ_{xy} are comparable to the measure $A \mapsto \int_A 1/|x-z| d\mathcal{H}^2(z)$. An elementary calculation gives

$$\left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^\infty(\mathcal{C}_{K_-}; \mu_{xy})} \lesssim 1.$$

An analogous argument proves the bound for \mathcal{C}_{K_+} .

Let ρ be admissible for Γ . Then

$$\begin{aligned} 1 &\leq \int_\Gamma \int_\gamma \rho ds d\sigma(\gamma) = \int_{\text{spt} \Gamma} \rho d\nu \\ &= \int_{S_{\mathbf{a}, M}} \rho \frac{d\nu}{d\mu_{xy}} d\mu_{xy} \leq \|\rho\|_{L^1(\mu_{xy})} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^\infty(\mu_{xy})} \lesssim \|\rho\|_{L^1(S_{\mathbf{a}, M}, \mu_{xy})}. \end{aligned}$$

Thus (2.1) holds. This completes the proof that $S_{\mathbf{a}}$ admits a 1-Poincaré inequality. \square

It remains to construct the block family described in the statement of Lemma 5.9.

Proof of Lemma 5.9. Recall that $m \leq M$ is chosen so that $10s_m \leq |x-y| < 10s_{m-1}$.

We construct the sequence of blocks and path families by induction. To make the proof more readable, we outline the basic steps, and leave the details to the reader. The basic idea is as follows: we use the expanding and turning lemmas (Lemmas 5.4–5.7) to build a sequence of blocks which grow in size at a linear rate as they travel away from x until reaching size $\sim |x-y|/100$. We do the same for y , and then join up the two sequences using the same lemmas. The procedure is akin to the construction of uniform curves inside uniform domains.

We now describe the construction in more detail, assuming (5.1) in order to simplify the argument.

First, $x \in T \subset T'$ for some $T \in \mathcal{T}_M$ and $T' \in \mathcal{T}_{M-1}$, and we can find a 1 by 2 (or 2 by 1) directed block (\mathcal{C}_0, L_0) so that $T \subset \mathcal{C}_0 \subset T'$, and L_0 is the short edge of \mathcal{C}_0 furthest from x , and L_0 does not meet the boundary of T' or the square of side s_M removed from T' in more than one point. This gives us our first directed block (\mathcal{C}_0, L_0) , which satisfies conditions (1) and (2), and we define Γ_0 according to condition (3).

The induction step is as follows. We assume that we have a sequence of blocks contained in a 1 by 2 (or 2 by 1) directed block (\mathcal{C}_-, L_-) on scale s_n , which is contained in some $T' \in \mathcal{T}_{n-1}$ in such a way that the short edge L_- does not meet the the boundary of T' , or of the central removed square of T' , in more than one point.

We choose a 1 by 2 (or 2 by 1) directed block (\mathcal{C}_+, L_+) on scale s_{n-1} so that $T' \subset \mathcal{C}_+$, and L_+ is the short edge of \mathcal{C}_+ furthest from \mathcal{C}_- , and L_+ does not meet the boundary or centrally removed square of the square $T'' \in \mathcal{T}_{s_{n-2}}$ with $T' \subset T''$.

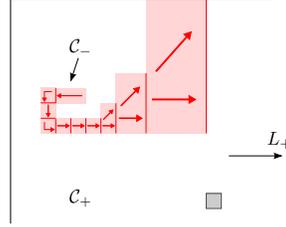


FIGURE 7. Steps (1) and (2) of the algorithm

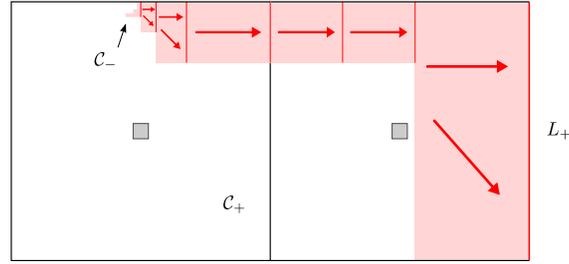


FIGURE 8. Steps (3)–(5) of the algorithm

We now build a sequence of directed blocks

$$(\mathcal{C}_-, L_-) = (\mathcal{C}'_0, L'_0), (\mathcal{C}'_1, L'_1), \dots, (\mathcal{C}'_t, L'_t)$$

inside \mathcal{C}_+ , where $L'_t = L_+$, and where for $i = 1, \dots, t$, (\mathcal{C}_i, L_i) follows $(\mathcal{C}_{i-1}, L_{i-1})$. Moreover, these blocks satisfy conditions (4),(5) and (7), and their diameters sum to $\asymp \text{dist}(\mathcal{C}_-, L_+)$.

The sequence of directed blocks is constructed using the following algorithm. See figures 7 and 8 for an illustration.

- (1) Use turning Lemma 5.6 and straight Lemma 5.7 between zero and six times to build a chain of blocks so that the last block is coherent with L_+ , closer to L_+ than \mathcal{C}_- , and is not contained in the central column of (\mathcal{C}_+, L_+) .
- (2) Use expanding Lemma 5.4 repeatedly to double away from the central column until, with an expansion by a factor between two and four, the long edge of \mathcal{C}_+ is reached.
- (3) Use expanding Lemma 5.4 repeatedly to double towards the central column, until of size $\geq \text{diam}(L_+)/5$.
- (4) Go straight (Lemma 5.7) until past the last removed square of the central column.
- (5) Expand by a factor less than five (Lemma 5.5), with the last edge L_+ .

We repeat this construction, growing in scale each time, until we are on scale s_m , then again until we have a block of size $|x - y|/100$, at a distance less than $|x - y|/10$ from x . This gives us most of the sequence of blocks with negative index.

We do the same for y , getting most of the (reverse) sequence of blocks with positive index, then join the two chains together using Lemmas 5.6 and 5.7. \square

6. VALIDITY OF THE POINCARÉ INEQUALITY: THE CASE $\mathbf{a} \in \ell^2 \setminus \ell^1$

In this section we address the case $\mathbf{a} \in \ell^2 \setminus \ell^1$. Our goal (Proposition 6.6) is to prove that in this case the carpet $S_{\mathbf{a}}$ verifies the p -Poincaré inequality for each $p > 1$.

The overall structure of the proof is similar to that in the previous section. According to Theorems 2.1 and 2.5, it suffices to verify the weighted modulus lower bound (2.1) on the precarpets $S_{\mathbf{a},M}$ with constants independent of M . To this end, we construct suitable curve families joining pairs of points in $S_{\mathbf{a},M}$ by concatenating curve families in suitable blocks. The construction of these curve families relies on ‘building block’ lemmas analogous to Lemmas 5.4–5.7 from the previous section. However, the proofs of the ‘building block’ lemmas in the cases $\mathbf{a} \in \ell^1$ and $\mathbf{a} \in \ell^2 \setminus \ell^1$ are quite different, and require two completely different methods.

Recall that in the case $\mathbf{a} \in \ell^1$, the thinnest part of the carpet has positive length, so we were able to use combinatorial constructions to build modular curve families avoiding all omitted squares in the precarpet $S_{\mathbf{a},M}$.

In the case $\mathbf{a} \in \ell^2$, considering only curves which pass through a thin part of the carpet cannot suffice. We instead construct a larger curve family using “bending” machinery which we will develop in subsection 6.2. We use this bending machinery to verify the p -Poincaré inequality for $p > 1$ via Keith’s theorem 2.1. (This machinery does not apply to the previous case, as the Hölder conjugate exponent q must be finite in order to use Proposition 6.11.)

The bending machinery, and hence our building block lemmas for $p > 1$, require the following condition:

$$(6.1) \quad a_i < a_{**} \quad \text{for all } i,$$

where $a_{**} \in (0, a_*)$ is a fixed, universal constant whose value is determined in subsection 6.2. Here as always we have $\mathbf{a} = (a_1, a_2, \dots) \in \ell^2$.

As before, Theorem 2.2 permits us to reduce to the case that (6.1) holds, because any $S_{\mathbf{a}}$ with $\mathbf{a} \in \ell^2$ is the finite union of similar copies of some carpet $S_{\mathbf{a}'}$, where $\mathbf{a}' \in \ell^2$ and all entries of \mathbf{a}' are less than a_{**} , glued along their boundaries. For the remainder of this section, we impose assumption (6.1).

Fix $p > 1$ with Hölder conjugate $q < \infty$. We first define a notion of q -connection which generalizes the previous notion of ∞ -connection. As we will employ bending machinery in this section rather than restricting to paths in the thin part of the carpet, the choice of the distinguished set π_M is simpler in this setting. For a directed block (\mathcal{C}, L) in a square T' , we let $\pi_M(L) = L$ and we let h be the corresponding bijection (which now works out to be the restriction of an affine map).

Definition 6.1. Suppose that the directed block (\mathcal{C}_2, L_2) follows the directed block (\mathcal{C}_1, L_1) , and let $h: \pi_M(L_1) \rightarrow \pi_M(L_2)$ be the natural ordered bijection. A path family Γ on $\pi_M(L_1)$ in \mathcal{C}_2 is called a q -connection from L_1 to L_2 through \mathcal{C}_2 (with constant C) if $\gamma(1) = h(\gamma(0))$ for each $\gamma \in \Gamma$, and if $\nu_\Gamma \ll \mathcal{H}^2 \llcorner \text{spt}(\Gamma)$ with

$$(6.2) \quad \left\| \frac{d\nu_\Gamma}{d\mathcal{H}^2} \right\|_{L^q(\mathcal{C}_2; \mathcal{H}^2)} \leq C (\mathcal{H}^1(\pi_M(L_1)))^{-1+2/q}.$$

The mysterious exponent on the right hand side of (6.2) can be justified by a dimensional analysis. Note that the measure ν_Γ is homogeneous of degree 1 relative to the scalings $x \mapsto \lambda x$, $\lambda > 0$, of \mathbb{R}^2 . Hence the Radon–Nikodym derivative $\frac{d\nu_\Gamma}{d\mathcal{H}^2}$ is homogeneous of degree -1 . Since the

L^q norm is computed with respect to Lebesgue measure, it follows that the left hand side of (6.2) is homogeneous of degree $-1 + \frac{2}{q}$.

In this setting of $\mathbf{a} \in \ell^2$, our building block lemmas take the following form. Recall that we fix $0 < m \leq M$, and are given directed blocks (\mathcal{C}_2, L_2) following (\mathcal{C}_1, L_1) in a directed square $(T', L') \in \mathcal{T}_{m-1}$.

Lemma 6.2 (Expanding). *Suppose that*

- (\mathcal{C}_1, L_1) and (\mathcal{C}_2, L_2) are coherent with (T', L') ,
- the sides of \mathcal{C}_2 perpendicular to L_2 have length equal to that of L_1 , and
- it holds that $\mathcal{H}^1(L_2)/\mathcal{H}^1(L_1) \leq 10$.

Then there is a q -connection Γ from L_1 to L_2 through \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_2)$.

Lemma 6.3 (Expanding to the parent generation). *Suppose that*

- (\mathcal{C}_1, L_1) is coherent with (T', L_2) , where $L_2 = L'$,
- the length of L_1 is equal to the length of an edge of \mathcal{C}_2 perpendicular to L_2 ,
- L_1 intersects an edge of T' perpendicular to L' , and
- it holds that $\mathcal{H}^1(L_2)/\mathcal{H}^1(L_1) \leq 10$.

Then there is a q -connection Γ from L_1 to L_2 through \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_2)$.

Lemma 6.4 (Turning). *Suppose that*

- L_1 and L_2 are perpendicular, and
- all edges of \mathcal{C}_2 have length equal to the length of L_1 .

Then there is a q -connection Γ from L_1 to L_2 through \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_2)$.

Lemma 6.5 (Going straight). *Suppose that*

- (\mathcal{C}_1, L_1) and (\mathcal{C}_2, L_2) are coherent with (T', L') ,
- L_1 and L_2 are of equal length, and
- the sides of \mathcal{C}_2 perpendicular to L_2 have length in $[\mathcal{H}^1(L_1)/2, 10\mathcal{H}^1(L_1)]$.

Then there is a q -connection Γ from L_1 to L_2 through \mathcal{C}_2 with constant $C = C(\|\mathbf{a}\|_2)$.

To produce the desired q -connection in these lemmas, we will use the bending machinery which we develop in subsection 6.2. We therefore postpone the proof of these lemmas until that time. Assuming for the moment their validity, we complete the proof of the p -Poincaré inequality.

6.1. Verification of the p -Poincaré inequality for $p > 1$. We prove the following proposition.

Proposition 6.6. *Suppose that $\mathbf{a} \in \ell^2$ and $p > 1$. Then $S_{\mathbf{a}}$ admits a p -Poincaré inequality.*

Proof. This proof is virtually identical to that of Proposition 5.8, using the new building block lemmas 6.2–6.5. Lemma 5.9 remains the same, except that condition (4) is replaced by

- (4') For each $i = (K_- + 1), \dots, -1$, (\mathcal{C}_i, L_i) follows $(\mathcal{C}_{i-1}, L_{i-1})$; for each $i = 1, \dots, K_+ - 1$, (\mathcal{C}_i, L_i) follows $(\mathcal{C}_{i+1}, L_{i+1})$; (\mathcal{C}_0, L_1) follows $(\mathcal{C}_{-1}, L_{-1})$, and (\mathcal{C}_0, L_{-1}) follows (\mathcal{C}_1, L_1) . In each case, Γ_i is a q -connection from L_{start} to L_{finish} through \mathcal{C}_i with constant C_0 . (See the remarks following Lemma 5.9 for the definitions of L_{start} and L_{finish} .)

This condition follows by using Lemmas 6.2–6.5 and exactly the same argument as before.

We now complete the proof of Proposition 6.6. First, concatenate the path families $\Gamma_{K_-}, \dots, \Gamma_{K_+}$ by gluing paths using the natural ordered bijection on each block. This gives a path family Γ consisting of pairwise disjoint, rectifiable curves joining x to y , carrying a probability measure $\sigma = \sigma_\Gamma$ on Γ which agrees with σ_{Γ_i} for each i on \mathcal{C}_i . The measure $\nu = \nu_\Gamma$ on the support of Γ defined as in (5.4) restricts to ν_{Γ_i} on each \mathcal{C}_i , and is absolutely continuous with respect to μ_{xy} . For $i = K_-, \dots, K_+$, we have the following bound on the Radon–Nikodym derivative $\frac{d\nu}{d\mu_{xy}}$:

$$\begin{aligned} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mathcal{C}_i; \mu_{xy})} &\asymp \left\| \frac{d\nu}{d\mathcal{H}^2} \right\|_{L^q(\mathcal{C}_i; \mathcal{H}^2)} \text{diam}(\mathcal{C}_i)^{1-1/q} \\ &\lesssim (\mathcal{H}^1(\pi_M(L_i)))^{-1+2/q} \text{diam}(\mathcal{C}_i)^{1-1/q} \\ &\lesssim \text{diam}(\mathcal{C}_i)^{-1+2/q} \text{diam}(\mathcal{C}_i)^{1-1/q} = \text{diam}(\mathcal{C}_i)^{1/q}. \end{aligned}$$

This bound also holds on \mathcal{C}_{K_-} and \mathcal{C}_{K_+} : note that on $\text{spt}(\Gamma_{K_-}) \subset \mathcal{C}_{K_-}$, both ν and μ_{xy} are comparable to the measure $A \mapsto \int_A 1/|x-z| d\mathcal{H}^2(z)$. An elementary calculation gives

$$\left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mathcal{C}_{K_-}; \mu_{xy})} \lesssim s_M^{1/q} \asymp \text{diam}(\mathcal{C}_{K_-})^{1/q}.$$

An analogous argument proves the bound for \mathcal{C}_{K_+} . Summing over i gives

$$\left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mu_{xy})}^q = \sum_{i=K_-}^{K_+} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mathcal{C}_i; \mu_{xy})}^q \lesssim \sum_{i=K_-}^{K_+} \text{diam}(\mathcal{C}_i) \lesssim |x-y|.$$

If ρ is admissible for Γ , then

$$\begin{aligned} 1 &\leq \int_\Gamma \int_\gamma \rho ds d\sigma(\gamma) = \int_{\text{spt} \Gamma} \rho d\nu \\ &= \int_{S_{\mathbf{a}, M}} \rho \frac{d\nu}{d\mu_{xy}} d\mu_{xy} \leq \|\rho\|_{L^p(\mu_{xy})} \left\| \frac{d\nu}{d\mu_{xy}} \right\|_{L^q(\mu_{xy})} \lesssim \|\rho\|_{L^p(\mu_{xy})} |x-y|^{1/q}. \end{aligned}$$

Consequently

$$\int_{S_{\mathbf{a}, M}} \rho^p d\mu_{xy} \gtrsim |x-y|^{-p/q} = |x-y|^{1-p}$$

and so (2.1) holds. This completes the proof of Proposition 6.6, modulo the building block lemmas 6.2–6.5. \square

6.2. Bending curve families. In this section, we introduce the bending machinery needed to prove Lemmas 6.2–6.5. The methods used allow us to build curve families of positive modulus in a wide class of compact planar sets, as illustrated by the following theorem, which gives a general sufficient condition for such curve families.

Theorem 6.7. *Let $D \subset \mathbb{R}^2$ be the closure of a domain and let $\mathbf{a} = (a_1, a_2, \dots) \in \ell^2$ with $a_m \in (0, 1)$ for all m . For each m , let $s_m = \prod_{j=1}^m a_j$, and let \mathcal{U}_m be a family of disjoint open subsets of D . Assume that the following two conditions are satisfied:*

- for all $U \in \mathcal{U}_m$, $\text{diam} U \leq 2s_m$, and

- for all $U \in \mathcal{U}_m$ and all $V \in \{\partial D\} \cup \mathcal{U}_1 \cup \mathcal{U}_2 \cup \dots \cup \mathcal{U}_m$ with $V \neq U$, we have $\text{dist}(U, V) \geq \frac{2}{5}s_{m-1}$.

Let $S_M := D \setminus \cup\{U : U \in \mathcal{U}_1 \cup \dots \cup \mathcal{U}_M\}$ and

$$S = \bigcap_{M \geq 0} S_M.$$

Then for all $p > 1$ and all relatively open balls $B \subset S$, there exists a curve family contained in B with positive p -modulus with respect to the measure \mathcal{H}^2 restricted to S .

The coefficients 2 and $\frac{2}{5}$ in Theorem 6.7 have been fixed for the sake of definiteness and can be varied without changing the result.

In the setting of the carpets $S_{\mathbf{a}}$, our arguments give the following corollary, independent of Theorem 1.6.

Corollary 6.8. *For any $p > 1$ and $\mathbf{a} \in \ell^2$, there exists a positive constant $C = C(p, \mathbf{a})$ so that the p -modulus of the curve family joining the left hand edge to the right hand edge of $S_{\mathbf{a}}$ is at least C . If \mathbf{a} is monotone decreasing, then $C = C(p, \|\mathbf{a}\|_2)$.*

The basic idea of the construction in this section is as follows. We present an algorithm which accepts as input a family of curves in the plane and which yields as output a new family of curves which avoids a prespecified obstacle at a small quantitative multiplicative cost to the p -modulus. We apply this algorithm recursively to avoid all of the omitted sets. The algorithm in question works by splitting the family of input curves in two pieces which are deformed to pass on either side of the obstacle. (Similar ideas appear in a paper of Chris Bishop [1] on A_1 deformations of the plane.)

The curve families that we consider are axiomatized in the following definition.

Definition 6.9. An open measured family of C^2 curves is a collection Γ of disjoint, oriented C^2 curves in a set $X \subset \mathbb{R}^2$, together with a probability measure σ on Γ , such that the union of all the curves in Γ , denoted $\text{spt } \Gamma$, is an open subset of X . We will denote such a pair by (Γ, σ) , or just by Γ if the measure σ is understood.

There is a natural measure $\nu_\Gamma = \nu_{(\Gamma, \sigma)}$ defined on $\text{spt } \Gamma$ by

$$(6.3) \quad \nu_\Gamma(V) = \int_\Gamma \mathcal{H}^1(V \cap \gamma) d\sigma(\gamma).$$

At this point, the integral in (6.3) should be interpreted as an upper integral with value in $[0, \infty]$. However, under the conditions of Definition 6.10, ν_Γ will be a finite Borel measure.

We assume that each curve in Γ is parameterized with nonzero speed, consistent with the specified orientation. Since each curve in Γ is C^2 , there is a vector field $\dot{\Gamma}$ defined on $\text{spt } \Gamma$ such that $\dot{\Gamma}(x)$ coincides with the unit tangent vector to the unique curve $\gamma_x \in \Gamma$ passing through x at time t_x . In fact,

$$(6.4) \quad \dot{\Gamma}(x) = \frac{\gamma'_x(t_x)}{|\gamma'_x(t_x)|}.$$

Definition 6.10. Fix $\delta_0 \geq 0$ and $r_0 > 0$. Suppose (Γ, σ) is an open measured family of C^2 curves, with ν_Γ defined as in (6.3). We say that (Γ, σ) is a δ_0 -good family of curves on scales less than r_0 if for any ball $B(z, r)$, $z \in X$, $0 < r \leq r_0$ we have the following properties:

FIGURE 9. Structure of $\overline{\text{spt } \Gamma} \cap B(z, r)$

- (A) If $B(z, r)$ does not contain any endpoint of any $\gamma \in \Gamma$, then the complement of the closure of $\text{spt } \Gamma$ in $B(z, r)$ is a connected open set.
 (B) For any $x, y \in B(z, r)$,

$$\angle(\dot{\Gamma}(x), \dot{\Gamma}(y)) \leq \delta_0 \left(\frac{|x - y|}{2r_0} \right)^{2/3},$$

where $\angle(\mathbf{v}, \mathbf{w})$ denotes the angle between vectors \mathbf{v} and \mathbf{w} .

- (C) There is a constant $A_{z,r} \in (0, \infty)$ so that on $B(z, r) \cap \text{spt } \Gamma$ the Radon-Nikodym derivative $w_\Gamma = d\nu_\Gamma/d\mathcal{H}^2$ exists, is $\frac{2}{3}$ -Hölder continuous with constant $(2r_0)^{-2/3}A_{z,r}\delta_0$, and satisfies

$$w_\Gamma(B(z, r) \cap \text{spt } \Gamma) \subset [(1 + \delta_0)^{-1}A_{z,r}, (1 + \delta_0)A_{z,r}].$$

The first condition ensures that $\overline{\text{spt } \Gamma} \cap B(z, r)$ is either the empty set, one half of the ball, all of the ball except for one open gap, or all of the ball. Figure 9 illustrates typical instances of this.

The last two conditions guarantee that the vector field $\dot{\Gamma}$ is $\frac{2}{3}$ -Hölder continuous (with suitable constant) and that the Radon-Nikodym derivative $w_\Gamma = \frac{d\nu_\Gamma}{d\mathcal{H}^2}$ exists and is locally close to constant on $\text{spt } \Gamma$.

Why are the vector fields only Hölder continuous? If they were Lipschitz continuous, then by the uniqueness of solutions to ODE with Lipschitz coefficients, the curves could not split to bend round an obstacle. The choice of $\frac{2}{3}$ is fixed in view of the cubic spline which we construct in Lemma 6.13(4). This choice is merely a convenience; any Hölder exponent strictly less than one would serve our purposes equally well.¹

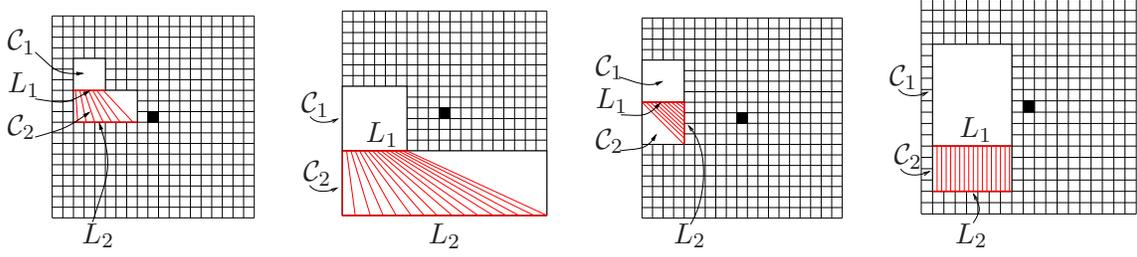
The following proposition provides the key inductive step in bending curve families with control on their modulus. We postpone its proof until subsection 6.4.

Proposition 6.11. *For any sufficiently small $\delta_0 > 0$, there exist positive constants a' and C with the following property.*

Suppose (Γ_i, σ_i) is a δ_0 -good family of curves on scales smaller than s_i in \mathbb{R}^2 , for some $s_i > 0$, and we are given $a_{i+1} \in (0, a']$. Let $\nu_i = \nu_{\Gamma_i}$ be the natural measure, let $w_i = d\nu_i/d\mathcal{H}^2$ be the corresponding weight, and set $s_{i+1} = a_{i+1}s_i$. Then given any $z \in \text{spt } \Gamma_i$, with no curves in Γ_i stopping inside $B(z, s_i)$, we can deform (Γ_i, σ_i) inside $B(z, s_i/5)$ into a new open measured curve family $(\Gamma_{i+1}, \sigma_{i+1})$ that is δ_0 -good on scales smaller than s_{i+1} , so that $\text{spt } \Gamma_{i+1}$ does not meet $B(z, 2s_{i+1})$ and

$$(6.5) \quad \int_{B(z, s_i/5)} |w_{i+1}|^q d\mathcal{H}^2 \leq (1 + Ca_{i+1}^2) \int_{B(z, s_i/5)} |w_i|^q d\mathcal{H}^2.$$

¹Note that we do not explicitly solve the ODE corresponding to the vector field $\dot{\Gamma}$. In the actual proof of Proposition 6.11, the Hölder continuity assumption in Definition 6.10(C) arises from Lemma 6.13.


 FIGURE 10. Initial curve families for the $p > 1$ case of Lemmas 5.4 - 5.7

By “deform”, we mean that there exists a C^2 homeomorphism of the plane which restricts to the identity outside $B(z, s_i/5)$, and, up to discarding finitely many curves, induces a well defined, measure preserving bijection between (Γ_i, σ_i) and $(\Gamma_{i+1}, \sigma_{i+1})$. We emphasize that the numbers $s_i > 0$ and $a_{i+1} \in (0, a']$ in the statement of Proposition 6.11 are arbitrary and are not assumed to be arising from a specific sequence \mathbf{a} under consideration.

Intuitively, Proposition 6.11 asserts that we can deform Γ_i inside a ball B on a given scale s_i so as to avoid a prespecified obstacle of size s_{i+1} (in this case, a ball of radius $2s_{i+1}$ concentric with B) and so that the ℓ^q norm of the associated weight increases multiplicatively by at most a factor of $1 + Ca_{i+1}^2$, where C is independent of \mathbf{a} . The point is that we can repeatedly apply the proposition (on smaller and smaller scales) without losing control of δ_0 .

6.3. Using the bending machinery. We now prove our building block lemmas, Theorem 6.7 and Corollary 6.8.

Proof of Lemmas 6.2–6.5. Recall that (C_2, L_2) is a directed block following (C_1, L_1) in a directed square $(T', L') \in \mathcal{T}_{m-1}$. Let Γ_m be the open, measured curve family consisting of straight line segments in \mathbb{R}^2 connecting each point of the interior of L_1 to the corresponding point of L_2 under the natural ordered bijection, equipped with the measure σ_m induced from normalized linear measure on L_1 . See Figure 10.

Let $w_m = d\nu_{\Gamma_m}/d\mathcal{H}^2$ be the natural weight function associated to Γ_m . Observe that on $\text{spt } \Gamma_m$, we have $\nu_{\Gamma_m} \asymp \mathcal{H}^1(\pi_M(L_1))^{-1}\mathcal{H}^2$, and that $\mathcal{H}^2(\text{spt } \Gamma_m) \asymp (\mathcal{H}^1(\pi_M(L_1)))^2$, so

$$(6.6) \quad \|w_m\|_q \asymp (\mathcal{H}^1(\pi_M(L_1)))^{-1+2/q}.$$

Since a q -connection must lie in $C_2 \cap S_{\mathbf{a}, M}$, we ‘bend’ this initial family around the subsquares of C_2 that were removed in the construction of $S_{\mathbf{a}, M}$. This construction is inductive, building open measured curve families (Γ_i, σ_i) for $m \leq i \leq M$.

Observe that there exists a universal constant $\delta_1 > 1$ so that any such initial curve family Γ_m is a δ_1 -good curve family on scales below s_m . We use the following sublemma.

Sublemma 6.12. *For any $\epsilon \in (0, 1)$, a δ_1 -good curve family on scales below r_1 is also a $(\delta_1\epsilon^{1/3})$ -good curve family on scales below ϵr_1 .*

Proof. The only non-trivial estimate is the last part of Definition 6.10(C). On a ball of radius ϵr_1 , with associated constant A , the ratio of maximum to minimum values of w is at most

$$\frac{(1 + \delta_1)^{-1}A + (2r_1)^{-2/3}A\delta_1(2\epsilon r_1)^{2/3}}{(1 + \delta_1)^{-1}A} \leq 1 + (1 + \delta_1)\delta_1\epsilon^{2/3} \leq (1 + \delta_1\epsilon^{1/3})^2. \quad \square$$

Fix the constant a_{**} in (6.1) so that

$$a_{**} = a' \delta_1^{-3} \delta_0^3 \leq a',$$

where δ_0 and a' are chosen by Proposition 6.11. The sublemma above shows that Γ_m is a δ_0 -good curve family on scales below $s'_m := \delta_1^{-3} \delta_0^3 s_m$.

The removed squares of side s_{m+1} in \mathcal{C}_2 are all at least $s_m/2$ apart, so we apply Proposition 6.11 to (Γ_m, σ_m) independently for some z in each such removed square, with values $s_i = s'_m$ and $s_{i+1} = s_{m+1}$; this is valid since $s_{m+1}/s'_m < a'$. Denote the resulting open, measured curve family by $(\Gamma_{m+1}, \sigma_{m+1})$, which is δ_0 -good on scales below s_{m+1} .

By applying (6.5) around each removed square, we have that the natural weight function w_{m+1} associated to Γ_{m+1} satisfies

$$(6.7) \quad \|w_{m+1}\|_q \leq (1 + C a_{m+1}^2)^{1/q} \|w_m\|_q.$$

Similarly, for $i = m+1, \dots, M-1$, we build $(\Gamma_{i+1}, \sigma_{i+1})$ from (Γ_i, σ_i) by applying Proposition 6.11 independently for each removed square of side s_{i+1} ; this is valid since $s_{i+1}/s_i = a_{i+1} \leq a_{**} < a'$. We obtain an open, measured curve family $(\Gamma, \sigma) = (\Gamma_M, \sigma_M)$, with $\text{spt } \Gamma$ in $\mathcal{C}_2 \cap S_{\mathbf{a}, M}$. Iterating the weight bound of (6.7), we see that

$$\left\| \frac{d\nu_\Gamma}{d\mathcal{H}^2} \right\|_{L^q(\mathcal{C}_2, \mathcal{H}^2)} = \|w_M\|_q \leq \left(\prod_{i=m+1}^M (1 + C a_i^2) \right)^{1/q} \|w_m\|_q \leq \exp\left(\frac{C}{q} \|\mathbf{a}\|_2^2\right) \mathcal{H}^1(\pi_M(L_1))^{-1+2/q},$$

where this last inequality uses (6.6). This implies that the output family is a q -connection. \square

A similar argument extends to give modulus bounds.

Proof of Theorem 6.7. We fix $a' < \frac{1}{100}$, δ_0 , and C as in Proposition 6.11 and choose M so that $a_i < a'$ when $i \geq M$. Note that if \mathbf{a} is monotone decreasing, then M depends only on $\|\mathbf{a}\|_2$.

If S has nonempty interior we are done, since open sets in \mathbb{R}^2 certainly contain curve families with positive p -modulus. Otherwise there exists $\mathcal{U}_{M'} \neq \emptyset$, for some $M' \geq M$. Choose $U \in \mathcal{U}_{M'}$ so that $\text{dist}(U, \partial D) \geq \frac{2}{5} s_{M'-1} \geq 10 s_{M'}$. We choose a square $W \subset D$ of side $s_{M'}$ so that $\text{dist}(W, U)$ is between $2 s_{M'}$ and $3 s_{M'}$. By the assumptions of the theorem, and choice of M , W only meets sets from $\cup_{m \geq (M'+1)} \mathcal{U}_m$, which have diameter much smaller than W .

We choose coordinates so that $W = [0, s_{M'}]^2$. Let $\Gamma_{M'}$ be the family of curves $\{\gamma_u : u \in (0, s_{M'})\}$, where $\gamma_u : (0, s_{M'}) \rightarrow W$ is defined as $\gamma_u(t) = (t, u)$. We equip $\Gamma_{M'}$ with the probability measure $\sigma_{M'}$ given by scaling Lebesgue measure on $(0, s_{M'})$ by $s_{M'}^{-1}$. Observe that $\Gamma_{M'}$ is a 0-good family of curves on scales below $s_{M'}$, with $w_{M'} \equiv s_{M'}^{-1}$.

We now build measured curve families (Γ_m, σ_m) for $m \geq M'$ that are δ_0 -good on scales below s_m , with the additional property that

$$(6.8) \quad \text{spt } \Gamma_m \cap ([s_{M'}/4, 3s_{M'}/4] \times \mathbb{R}) \subset S_m.$$

Let $\nu_m = \nu_{\Gamma_m}$ be the natural measure on $\text{spt } \Gamma_m$ and let $w_m = d\nu_m/d\mathcal{H}^2$ be the corresponding weight.

The construction is inductive. Assume that we have constructed a measured curve family (Γ_m, σ_m) that is δ_0 -good on scales below s_m . We apply Proposition 6.11 to bend Γ_m around each set $V \in \mathcal{U}_{m+1}$ which meets $[s_{M'}/8, 7s_{M'}/8] \times [-s_{M'}, 2s_{M'}]$. As the sets in \mathcal{U}_{m+1} are all at least s_m apart, we can apply Proposition 6.11 at each location independently to create a new measured

curve family $(\Gamma_{m+1}, \sigma_{m+1})$, which is δ_0 -good on scales below s_{m+1} . (Proposition 6.11 requires that no curve ends near where we bend; this is why we restrict the obstacles that we bend around.) These curve families satisfy (6.8) for $m \geq M'$.

As in the proof of Lemmas 6.2–6.5, we conclude that for all $m \geq M'$ the natural weight function w_m associated to Γ_m satisfies

$$(6.9) \quad \|w_m\|_q \leq \left(\prod_{i=M'+1}^{\infty} (1 + Ca_i^2) \right)^{1/q} \|w_{M'}\|_q \leq \exp\left(\frac{C}{q} \|\mathbf{a}\|_2^2\right) \|w_{M'}\|_q$$

If $\rho : B \rightarrow [0, \infty]$ is admissible for $\text{mod}_p \Gamma_m$, that is, $1 \leq \int_{\gamma} \rho d\mathcal{H}^1$ for each $\gamma \in \Gamma_m$, then by averaging over Γ_m with respect to σ_m , we see that

$$(6.10) \quad 1 \leq \int_{\Gamma_m} \int_{\gamma} \rho d\mathcal{H}^1 d\sigma_m(\gamma) = \int_{\text{spt} \Gamma_m} \rho d\nu_m = \int_{\text{spt} \Gamma_m} \rho w_m d\mathcal{H}^2 \leq \|\rho\|_p \|w_m\|_q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Combining (6.9), (6.10) and $\mathbf{a} \in \ell^2$, we see that $\text{mod}_p \Gamma_m$ is uniformly bounded from below independent of m .

Each curve $\gamma \in \Gamma_m$ has a subcurve γ' which joins the set $\{s_{M'}/4\} \times \mathbb{R}$ to $\{3s_{M'}/4\} \times \mathbb{R}$, and $\gamma' \subset S_m$. Let Γ'_m be the collection of all such curves. By basic properties of the modulus, $\text{mod}_p \Gamma'_m \geq \text{mod}_p \Gamma_m$.

Thus for every $m \geq M'$, there is a curve family Γ'_m in S_m with p -modulus bounded from below independent of m . By the upper semicontinuity of modulus (see, for instance, Heinonen–Koskela [19, §3] or Keith [22, Theorem 1]), this bound will continue to the limit. \square

Proof of Corollary 6.8. Again, choose a' as in Proposition 6.11 and choose M so that $a_i < a'$ when $i \geq M$. The p -modulus of the family of curves joining the left and right edges of the carpet $S_{\mathbf{a}}$ is bounded from below by the p -modulus of the family of curves Γ joining the left and right edges of the strip $([0, 1] \times [0, s_M]) \cap S_{\mathbf{a}}$.

Let Γ_M be the family of horizontal lines in $T = [0, 1] \times [0, s_M]$, with induced natural weight w_M , equal to s_M^{-1} on T , and so having $\|w_M\|_q = s_M^{-1/p}$. The argument in the proof of Theorem 6.7, in particular (6.9) and (6.10), give that any admissible function ρ for Γ satisfies

$$1 \leq \|\rho\|_p \exp\left(\frac{C}{q} \|\mathbf{a}\|_2^2\right) \|w_M\|_q.$$

Therefore, as $\|w_M\|_q \leq C(\mathbf{a}) < \infty$, we have $\text{mod}_p(\Gamma) > 0$. Finally, we note that in the case when \mathbf{a} is monotone decreasing, then M can be chosen only depending on $\|\mathbf{a}\|_2$ (and not on the actual sequence \mathbf{a}). This implies that $\|w_M\|_q \leq C(\|\mathbf{a}\|_2)$, and this establishes the final claim of the corollary. \square

It remains to establish Proposition 6.11. This is the goal of the following subsection. The argument is rather technical although essentially elementary. The reader is invited to skip the remainder of this section on a first reading of the paper.

6.4. Compressing curve families: the proof of Proposition 6.11. The following construction is standard. For the convenience of the reader we provide a short proof.

Lemma 6.13. *There is a C^2 function $\varphi : [-1, 1] \rightarrow [0, 1]$ which satisfies*

$$(1) \quad \varphi|_{[-0.1, 0.1]} \equiv 1,$$

- (2) *the support of φ lies in $[-0.9, 0.9]$,*
- (3) $|\varphi'| \leq 5/2$, $|\varphi''| \leq 25/2$, *and*
- (4) $|\varphi'| \leq 14\varphi^{2/3}$.

Proof. Choose $\varphi'' : [-1, 1] \rightarrow \mathbb{R}$ to be the simplest piecewise linear function whose graph passes through the points $(\pm 1, 0)$, $(\pm 0.9, 0)$, $(\pm 0.7, b)$, $(\pm 0.3, -b)$, $(\pm 0.1, 0)$, and $(0, 0)$, where $b > 0$ is a constant to be determined.

Assuming that $\varphi'(-1) = \varphi(-1) = 0$, we integrate to find φ . Note that $\varphi|_{[-0.1, 0.1]} \equiv 0.08b$, so we choose $b = 1/0.08 = 25/2$. With this choice, $|\varphi'| \leq 0.2b = 5/2$ and $|\varphi''| \leq b = 25/2$. Hence conditions (1), (2) and (3) are satisfied.

Finally, note that for $0 \leq h \leq 0.2$, $\varphi''(-0.9 + h) = 5bh = \frac{125}{2}h$, $\varphi'(-0.9 + h) = \frac{5}{2}bh^2 = \frac{125}{4}h^2$ and $\varphi(-0.9 + h) = \frac{5}{6}bh^3 = \frac{125}{12}h^3$, so

$$|\varphi'(-0.9 + h)| = 5 \left(\frac{3}{2} \right)^{2/3} |\varphi(-0.9 + h)|^{2/3} \leq 10 |\varphi(-0.9 + h)|^{2/3}.$$

This bound also applies for $x \in [0.7, 0.9]$. On the other hand, for $x \in [-0.7, 0.7]$ we have $\varphi(x) \geq \varphi(-0.7) = \frac{1}{12}$ and $|\varphi'(x)| \leq \frac{5}{2}$, so $|\varphi'(x)| \leq \frac{5}{2} = \frac{5}{2} \cdot 12^{2/3} \left(\frac{1}{12} \right)^{2/3} \leq 14 |\varphi(x)|^{2/3}$. \square

We now begin the proof of Proposition 6.11.

Proof of Proposition 6.11. We fix a positive constant $\delta_0 \leq \frac{1}{200}$. We will choose a large positive integer $N = N(\delta_0) \geq 5$; the precise choice will be made later in the proof. Finally, we assume that $a' \leq 10^{-2-N}$; we only consider $a_i < a'$.

Let (Γ_i, σ_i) be a δ_0 -good family of curves on scales smaller than s_i , let ν_i be the corresponding measure as defined in (6.3), and suppose that $z \in \text{spt } \Gamma_i$. We can apply an isometry of \mathbb{R}^2 to reduce to the case when $z = 0$,

$$\begin{aligned} B(z, 2s_{i+1}) \cap P &:= [-2s_{i+1}, 2s_{i+1}]^2 \\ &\subset Q := [-10^N s_{i+1}, 10^N s_{i+1}]^2 \\ &\subset R := [-s_i/10, s_i/10]^2 \subset B(z, s_i/5), \end{aligned}$$

and $\dot{\Gamma}_i(0)$ has horizontal slope. This last assertion, in conjunction with Definition 6.10(B), implies that all curves in R have slopes within $(1/5)\delta_0$ of zero. In particular, each curve $\gamma \in \Gamma_i$ is a graph over the x -axis inside R . Henceforth we will assume that each curve is given in graph form: $y = \gamma(x)$. Nevertheless, we continue to denote by $\gamma = \{(x, \gamma(x))\}$ the graph itself.

We choose a curve γ_0 which passes near P and either bounds an existing gap in Q , or is far from an existing gap in Q . To be precise, let U be the complement of $\overline{\text{spt } \Gamma_i}$ in $B(z, s_i/5)$ which, by condition (A), is a connected open set. Moreover, $\partial_\infty U \cap B(z, s_i/5)$ lies in one or two C^2 curves whose slopes satisfy, along with Γ_i , condition (B). If U meets $L = \{0\} \times [-3s_{i+1}, 3s_{i+1}]$, choose γ_0 which bounds an edge of U meeting L (as $(0, 0) \notin U$, such a γ_0 exists). If U does not meet L , choose $\gamma_0 \in \Gamma_i$ which passes through $(0, -3s_{i+1})$ or $(0, 3s_{i+1})$, chosen so that

$$(6.11) \quad \text{dist}(\gamma_0(0), U) \geq 5s_{i+1}.$$

(Recall that γ_0 is a graph over the x -axis and we have normalized so that $z = 0$.)

We will compress the curves inside Q into the complement of P , leaving everything unchanged in $R \setminus Q$. To build Γ_{i+1} we will delete γ_0 from Γ_i if necessary, and apply a diffeomorphism on $X \setminus \gamma_0$

to compress the remaining curves around P . The two options in the choice of γ_0 correspond to either enlarging an existing gap in Γ_i , or creating a new gap at least $4s_{i+1}$ from any previous gap.

We rescale the function φ from Lemma 6.13 to the scale of Q by defining

$$\tilde{\varphi}(x) = 6s_{i+1}\varphi\left(\frac{x}{10^N s_{i+1}}\right).$$

Note that $|\tilde{\varphi}'| \leq 10^{2-N}$, $|\tilde{\varphi}''| \leq 10^{2-2N} s_{i+1}^{-1}$, and

$$(6.12) \quad |\tilde{\varphi}'| \leq 10^{2-N} \left(\frac{\tilde{\varphi}}{2s_{i+1}}\right)^{2/3}.$$

We now define the *local compression map* $H : Q \setminus \{\gamma_0\} \rightarrow Q$. Let $g : Q \setminus \{\gamma_0\} \rightarrow [-1, 1]$ be given by

$$g(x, y) = \begin{cases} \varphi\left(\frac{y-\gamma_0(x)}{10^N s_{i+1}-\gamma_0(x)}\right) & \text{if } y \in (\gamma_0(x), 10^N s_{i+1}) \\ -\varphi\left(\frac{\gamma_0(x)-y}{10^N s_{i+1}+\gamma_0(x)}\right) & \text{if } y \in (-10^N s_{i+1}, \gamma_0(x)). \end{cases}$$

Since the functions γ_0 and φ are C^2 and $10^N s_{i+1} - \gamma_0$ and $10^N s_{i+1} + \gamma_0$ take values in $[0.99 \cdot 10^N s_{i+1}, 1.01 \cdot 10^N s_{i+1}]$, g is C^2 . The function g varies from 0 near the top of Q to 1 just above γ_0 , and from -1 just below γ_0 to 0 near the bottom of Q . Next let

$$h(x, y) = y + \tilde{\varphi}(x) \cdot g(x, y),$$

and define

$$H(x, y) = (x, h(x, y)).$$

Both h and H are C^2 , moreover, H is a diffeomorphism onto its image.

Extend g to be zero outside Q and extend H to be the identity outside Q . The new collection of curves is defined by pushing forward by the local compression map H :

$$(6.13) \quad \Gamma_{i+1} = \{H(\gamma) : \gamma \in \Gamma_i, \gamma \neq \gamma_0\}.$$

The probability measure σ_i on Γ_i pushes forward in the obvious way to a probability measure on Γ_{i+1} that we denote by σ_{i+1} . We define $\dot{\Gamma}_{i+1}$ and ν_{i+1} as in (6.4) and (6.3). Since $H(Q \setminus \{\gamma_0\})$ and P are disjoint, so are $\text{spt } \Gamma_{i+1}$ and P .

Proposition 6.14. $(\Gamma_{i+1}, \sigma_{i+1})$ is a δ_0 -good family of curves on scales below s_{i+1} , and $\|w_{i+1}\|_q \leq 2\|w_i\|_q$ on Q .

Assuming for the moment the validity of Proposition 6.14 we quickly complete the proof of Proposition 6.11. By condition (C), we have, for $A = A_{z, s_i}$,

$$(6.14) \quad \int_Q w_i^q d\mathcal{H}^2 \leq (1 + \delta_0)^q A^q \mathcal{H}^2(Q) \leq (1 + \delta_0)^q A^q \cdot 10^{2N} s_{i+1}^2.$$

As $z \in \text{spt } \Gamma_i$, by condition (A), we know that $\mathcal{H}^2(\text{spt } \Gamma_i \cap R) \geq \frac{1}{3} \mathcal{H}^2(R) = 75^{-1} s_i^2$. So we bound

$$(6.15) \quad \int_R w_i^q d\mathcal{H}^2 \geq (1 + \delta_0)^{-q} A^q \mathcal{H}^2(\text{spt } \Gamma_i \cap R) \geq 75^{-1} (1 + \delta_0)^{-q} A^q s_i^2.$$

Note $w_{i+1} = w_i$ on $R \setminus Q$, and Proposition 6.14 controls w_{i+1} on Q . Therefore, by (6.14) and (6.15) we have

$$\begin{aligned} \int_R w_{i+1}^q d\mathcal{H}^2 &\leq \int_{R \setminus Q} w_i^q d\mathcal{H}^2 + 2^q \int_Q w_i^q d\mathcal{H}^2 \\ &\leq \int_R w_i^q d\mathcal{H}^2 + C a_{i+1}^2 \int_R w_i^q d\mathcal{H}^2 \\ &= (1 + C a_{i+1}^2) \int_R w_i^q d\mathcal{H}^2, \end{aligned}$$

where $C = 75 \cdot 2^q (1 + \delta_0)^{2q} 10^{2N}$. This completes the proof of Proposition 6.11. \square

The proof of Proposition 6.14 is divided into three lemmas. In all three of these lemmas, the context is the modified measured curve family $(\Gamma_{i+1}, \sigma_{i+1})$ defined in (6.13).

Lemma 6.15. *Condition (A) of Definition 6.10 is satisfied.*

Proof. Recall that the curve γ_0 was chosen in one of two ways. First, suppose that γ_0 was chosen to contain part of the boundary of $U = B(z, s_i/5) \setminus \overline{\text{spt} \Gamma_i}$. It is clear that the deformation H has only enlarged this set, and it is easy to see that Γ_{i+1} will satisfy condition (A).

Now in the remainder of this proof, we suppose that $\gamma_0 \in \Gamma_i$ was chosen so that $\text{dist}(\gamma_0(0), U) \geq 5s_{i+1}$. We must check that the new open set opened up along γ_0 will not result in two open gaps in $\text{spt} \Gamma_{i+1}$ in a common s_{i+1} -ball.

Denote the curve which bounds the edge of U closest to γ_0 by γ_1 . Without loss of generality, we may assume that $\gamma_0(x) \leq \gamma_1(x)$ for all $x \in I := [-10^N s_{i+1}, 10^N s_{i+1}]$. To complete the proof of this lemma, it suffices to show that $\gamma_1(x) - \gamma_0(x) \geq 4s_{i+1}$ for all $x \in I$, since then in the image they will remain sufficiently far apart.

Let σ_0 be the σ_i measure of those curves of Γ_i which lie between γ_0 and γ_1 . For $x_1 \leq x_2$, let

$$T[x_1, x_2] = \{(x, y) \in Q \cap \text{spt} \Gamma_i : x_1 \leq x \leq x_2, \gamma_0(x) < y < \gamma_1(x)\}.$$

By (6.3), for any $x \in I$, $h > 0$ we have $\sigma_0 h \leq \nu_i(T[x, x+h])$. On the other hand, by condition (C) we have $\nu_i(T[x, x+h]) \leq (\gamma_1(x) - \gamma_0(x) + h/100)h(1 + \delta_0)A$, for the appropriate value of the constant $A = A_{z,r}$. Combining these and letting $h \rightarrow 0$, we see that

$$(6.16) \quad \sigma_0 \leq (1 + \delta_0)A(\gamma_1(x) - \gamma_0(x)).$$

Likewise, by considering $T[0, h]$, we see that

$$(\gamma_1(0) - \gamma_0(0) - h/100)h(1 + \delta_0)^{-1}A \leq \nu_i(T[0, h]) \leq 1.01\sigma_0 h,$$

and therefore

$$(6.17) \quad (\gamma_1(0) - \gamma_0(0))(1 + \delta_0)^{-1}A \leq 1.01\sigma_0.$$

Combining (6.11), (6.16) and (6.17), we see that

$$5s_{i+1} \leq \gamma_1(0) - \gamma_0(0) \leq (1 + \delta_0)A^{-1}1.01\sigma_0 \leq (1 + \delta_0)^2 1.01(\gamma_1(x) - \gamma_0(x)).$$

Therefore γ_0 and γ_1 are always at least $4s_{i+1}$ apart in Q . \square

Lemma 6.16. *Condition (B) of Definition 6.10 is satisfied.*

Proof. Let us consider $u_1, u_2 \in R \cap (\text{spt } \Gamma_{i+1})$ so that $|u_1 - u_2| \leq 2s_{i+1}$. Note that $u_k = H(v_k)$ for some $v_k \in \text{spt } \Gamma_i \cap R$, $k \in \{1, 2\}$. If v_1 and v_2 lie on different sides of γ_0 then one can calculate that $|v_1 - v_2| \leq 1.02|u_1 - u_2|$, as v_k and u_k lie on the same vertical line and the slope of γ_0 is close to zero. When v_1 and v_2 are on the same side of γ_0 , the same estimate follows from (6.28) below. Write

$$v_k = (x_k, \gamma_k(x_k)) = (\text{id} \otimes \gamma_k)(x_k)$$

for some $\gamma_k \in \Gamma_i$. We calculate the differential of the function $h \circ (\text{id} \otimes \gamma_k)$ as follows:

$$(6.18) \quad (h \circ (\text{id} \otimes \gamma_k))'(x_k) = C_{1k} + C_{2k} + C_{3k},$$

where $C_{1k} = \gamma'_k(x_k)$, $C_{2k} = \tilde{\varphi}'(x_k)g(v_k)$, and

$$C_{3k} = \tilde{\varphi}(x_k)(g \circ (\text{id} \otimes \gamma_k))'(x_k).$$

Eventually, we want to estimate the difference between $(h \circ (\text{id} \otimes \gamma_1))'(x_1)$ and $(h \circ (\text{id} \otimes \gamma_2))'(x_2)$. In view of (6.18), we write ΔC_1 , ΔC_2 , and ΔC_3 for the differences of the summands. We will estimate everything in terms of the scale-invariant quantity

$$(6.19) \quad \alpha(u_1, u_2) := \left(\frac{|u_1 - u_2|}{2s_{i+1}} \right)^{2/3}.$$

To estimate ΔC_1 we use the following elementary fact: for any $v, w \in [-1/4, 1/4]$, the vectors $\mathbf{v} = (1, v)$ and $\mathbf{w} = (1, w)$ satisfy

$$(6.20) \quad \frac{1}{2}|v - w| \leq \left| |\mathbf{v}|^{-1}\mathbf{v} - |\mathbf{w}|^{-1}\mathbf{w} \right| \leq 2|v - w|,$$

as one quickly sees from the identity

$$\frac{||\mathbf{v}|^{-1}\mathbf{v} - |\mathbf{w}|^{-1}\mathbf{w}|^2}{|v - w|^2} = \frac{2}{(1 + v^2)(1 + w^2) + (1 + vw)\sqrt{1 + v^2}\sqrt{1 + w^2}}.$$

Recall that $\dot{\Gamma}_i(v_k) = \frac{(1, \gamma'_k(x_k))}{\sqrt{1 + \gamma'_k(x_k)^2}}$ for $k \in \{1, 2\}$. Since $\dot{\Gamma}_i(v_k)$ is a unit vector,

$$(6.21) \quad |\dot{\Gamma}_i(v_1) - \dot{\Gamma}_i(v_2)| = 2 \sin \left(\frac{1}{2} \angle(\dot{\Gamma}_i(v_1), \dot{\Gamma}_i(v_2)) \right) \leq \angle(\dot{\Gamma}_i(v_1), \dot{\Gamma}_i(v_2)).$$

An application of (6.20) gives

$$(6.22) \quad \begin{aligned} |\Delta C_1| &= |\gamma'_1(x_1) - \gamma'_2(x_2)| \leq 2|\dot{\Gamma}_i(v_1) - \dot{\Gamma}_i(v_2)| \leq 2\angle(\dot{\Gamma}_i(v_1), \dot{\Gamma}_i(v_2)) \\ &\leq 2\delta_0 \left(\frac{|v_1 - v_2|}{2s_i} \right)^{2/3} \leq 3\delta_0 a_{i+1}^{2/3} \alpha(u_1, u_2), \end{aligned}$$

where $\alpha(u_1, u_2)$ is defined as in (6.19).

If one of the points v_1 and v_2 lies outside Q , then both are close to the edge of Q , where H is the identity, thus

$$(6.23) \quad |\Delta C_2| = |\Delta C_3| = 0.$$

We therefore assume that both v_1 and v_2 are in Q .

Suppose v_1 and v_2 lie on opposite sides of γ_0 . Since $|v_1 - v_2| \leq 3s_{i+1}$, we have that $|\tilde{\varphi}(x_1)| + |\tilde{\varphi}(x_2)| \leq 2|u_1 - u_2|$. Therefore, using (6.12) we see that

$$(6.24) \quad \begin{aligned} |\Delta C_2| &= |\tilde{\varphi}'(x_1)g(v_1) - \tilde{\varphi}'(x_2)g(v_2)| \leq |\tilde{\varphi}'(x_1)| + |\tilde{\varphi}'(x_2)| \\ &\leq 10^{2-N} \left(\frac{\tilde{\varphi}(x_1)}{2s_{i+1}} \right)^{2/3} + 10^{2-N} \left(\frac{\tilde{\varphi}(x_2)}{2s_{i+1}} \right)^{2/3} \\ &\leq 4 \cdot 10^{2-N} \alpha(u_1, u_2). \end{aligned}$$

Since v_1 and v_2 are both close to γ_0 , they are both in the region where $|g| = 1$, whence

$$(6.25) \quad \Delta C_3 = 0.$$

It remains to bound ΔC_2 and ΔC_3 when v_1 and v_2 lie on the same side of γ_0 . Without loss of generality, we may assume that both v_1 and v_2 are above γ_0 . Then $g(v_k) = \varphi(A_k/B_k)$ where $A_k = \gamma_k(x_k) - \gamma_0(x_k)$ and $B_k = 10^N s_{i+1} - \gamma_0(x_k)$. Note that

$$\begin{aligned} |A_k| &\leq 1.01 \cdot 10^N s_{i+1}, \quad |B_k| \in [0.99 \cdot 10^N s_{i+1}, 1.01 \cdot 10^N s_{i+1}], \\ |A_1 - A_2| &\leq 2|v_1 - v_2|, \quad |B_1 - B_2| \leq \frac{1}{100}|v_1 - v_2|. \end{aligned}$$

To estimate $|g(v_1) - g(v_2)|$ we use the simple estimate

$$(6.26) \quad \left| \frac{A_1}{B_1} - \frac{A_2}{B_2} \right| \leq \frac{|A_1 - A_2| \cdot |B_2| + |A_2| \cdot |B_1 - B_2|}{|B_1 B_2|} \leq 3 \cdot 10^{-N} \left(\frac{|v_1 - v_2|}{s_{i+1}} \right).$$

Thus

$$(6.27) \quad |g(v_1) - g(v_2)| = \left| \varphi \left(\frac{A_1}{B_1} \right) - \varphi \left(\frac{A_2}{B_2} \right) \right| \leq \|\varphi'\|_\infty \left| \frac{A_1}{B_1} - \frac{A_2}{B_2} \right| \leq 10^{1-N} \left(\frac{|v_1 - v_2|}{s_{i+1}} \right).$$

Since $u_1 - u_2 = v_1 - v_2 + (0, \tilde{\varphi}(x_1)g(v_1) - \tilde{\varphi}(x_2)g(v_2))$, the estimate $|u_1 - u_2| \geq 0.99|v_1 - v_2|$ follows from the following bound.

$$(6.28) \quad \begin{aligned} |\tilde{\varphi}(x_1)g(v_1) - \tilde{\varphi}(x_2)g(v_2)| &\leq |\tilde{\varphi}(x_1) - \tilde{\varphi}(x_2)| \|g\|_\infty + \|\tilde{\varphi}\|_\infty |g(v_1) - g(v_2)| \\ &\leq 10^{2-N} |x_1 - x_2| + 6s_{i+1} \cdot 10^{1-N} \left(\frac{|v_1 - v_2|}{s_{i+1}} \right) \leq 10^{3-N} |v_1 - v_2|. \end{aligned}$$

Observe too that

$$(6.29) \quad |\tilde{\varphi}'(x_1) - \tilde{\varphi}'(x_2)| \leq \|\tilde{\varphi}''\|_\infty |x_1 - x_2| \leq 10^{2-2N} s_{i+1}^{-1} |v_1 - v_2| \leq 10^{3-2N} \alpha(u_1, u_2).$$

From (6.27) we have $|g(v_1) - g(v_2)| \leq 10^{2-N} \alpha(u_1, u_2)$, which combines with (6.29) to get

$$(6.30) \quad \begin{aligned} |\Delta C_2| &\leq |\tilde{\varphi}'(x_1)| |g(v_1) - g(v_2)| + |\tilde{\varphi}'(x_1) - \tilde{\varphi}'(x_2)| |g(v_2)| \\ &\leq 10^{2-N} \cdot 10^{2-N} \alpha(u_1, u_2) + 10^{3-2N} \alpha(u_1, u_2) \leq 10^{5-2N} \alpha(u_1, u_2). \end{aligned}$$

Finally, we must bound $|\Delta C_3|$. As v_1, v_2 both lie above γ_0 , we have

$$(g \circ (\text{id} \otimes \gamma_k))'(x_k) = E_k \varphi'(A_k/B_k),$$

where A_k and B_k are as defined above and

$$E_k = \frac{A'_k B_k - A_k B'_k}{B_k^2}, \quad A'_k = \gamma'_k(x_k) - \gamma'_0(x_k), \quad B'_k = \gamma'_0(x_k).$$

Now $\max\{|A'_k|, |B'_k|\} \leq \frac{1}{100}$, so $|E_k| \leq 3 \cdot 10^{-2-N} s_{i+1}^{-1}$ and $(g \circ (\text{id} \otimes \gamma_k))'(x_k) \leq 10^{-1-N} s_{i+1}^{-1}$. Since v_1 and v_2 lie on the same side of γ_0 , we have

$$\max\{|A_1 - A_2|, |B_1 - B_2|\} \leq 10s_{i+1}\alpha(u_1, u_2) \quad \text{and} \quad \max\{|A'_1 - A'_2|, |B'_1 - B'_2|\} \leq \alpha(u_1, u_2),$$

so

$$\begin{aligned} |E_1 - E_2| &= \left| \frac{A'_1 B_1 - A_1 B'_1}{B_1^2} - \frac{A'_2 B_2 - A_2 B'_2}{B_2^2} \right| \\ &\leq 10^{1-4N} s_{i+1}^{-4} |(A'_1 B_1 - A_1 B'_1) B_2^2 - (A'_2 B_2 - A_2 B'_2) B_1^2| \\ &= 10^{1-4N} s_{i+1}^{-4} |B_1 B_2^2 (A'_1 - A'_2) + A'_2 B_1 B_2 (B_2 - B_1) + A_2 B_1^2 (B_2 - B_1) \\ &\quad + B'_1 B_1^2 (A_2 - A_1) + A_1 B'_1 (B_1 + B_2) (B_1 - B_2)| \\ &\leq 10^{1-4N} s_{i+1}^{-4} (2 \cdot 10^{1+3N} s_{i+1}^3 \alpha(u_1, u_2) + 3 \cdot 10^{2N} s_{i+1}^3 \alpha(u_1, u_2)) \leq 10^{3-N} s_{i+1}^{-1} \alpha(u_1, u_2). \end{aligned}$$

Thus $|(g \circ (\text{id} \otimes \gamma_1))'(x_1) - (g \circ (\text{id} \otimes \gamma_2))'(x_2)|$ is equal to

$$\begin{aligned} \left| E_1 \varphi' \left(\frac{A_1}{B_1} \right) - E_2 \varphi' \left(\frac{A_2}{B_2} \right) \right| &\leq |E_1 - E_2| \cdot \left| \varphi' \left(\frac{A_1}{B_1} \right) \right| + |E_2| \cdot \left| \varphi' \left(\frac{A_1}{B_1} \right) - \varphi' \left(\frac{A_2}{B_2} \right) \right| \\ &\leq |E_1 - E_2| \cdot \|\varphi'\|_\infty + |E_2| \cdot \|\varphi''\|_\infty \cdot \left| \frac{A_1}{B_1} - \frac{A_2}{B_2} \right| \\ &\leq 10^{4-N} s_{i+1}^{-1} \alpha(u_1, u_2) + 10^{1-2N} s_{i+1}^{-1} \alpha(u_1, u_2) \leq 10^{5-N} s_{i+1}^{-1} \alpha(u_1, u_2), \end{aligned}$$

while

$$|\tilde{\varphi}(x_1) - \tilde{\varphi}(x_2)| \leq \|\tilde{\varphi}'\|_\infty |x_1 - x_2| \leq 10^{2-N} |v_1 - v_2|.$$

Putting all this together,

$$\begin{aligned} |\Delta C_3| &\leq |\tilde{\varphi}(x_1)| |(g \circ (\text{id} \otimes \gamma_1))'(x_1) - (g \circ (\text{id} \otimes \gamma_2))'(x_2)| \\ &\quad + |\tilde{\varphi}(x_1) - \tilde{\varphi}(x_2)| |(g \circ (\text{id} \otimes \gamma_2))'(x_2)| \\ (6.31) \quad &\leq 6s_{i+1} \cdot 10^{5-N} s_{i+1}^{-1} \alpha(u_1, u_2) + 10^{2-N} |v_1 - v_2| \cdot 10^{-1-N} s_{i+1}^{-1} \\ &\leq 10^{6-N} \alpha(u_1, u_2) + 10^{2-2N} \alpha(u_1, u_2) \leq 10^{7-N} \alpha(u_1, u_2). \end{aligned}$$

We can now tie all these estimates together. Using (6.21) again, we estimate

$$\begin{aligned} \angle(\dot{\Gamma}_{i+1}(u_1), \dot{\Gamma}_{i+1}(u_2)) &\leq \frac{\pi}{2} |\dot{\Gamma}_{i+1}(u_1) - \dot{\Gamma}_{i+1}(u_2)| \\ &\leq \pi |(h \circ (\text{id} \otimes \gamma_1))'(x_1) - (h \circ (\text{id} \otimes \gamma_2))'(x_2)|. \end{aligned}$$

This follows from (6.20) upon noting that

$$\dot{\Gamma}_{i+1}(u_k) = (H \circ (\text{id} \otimes \gamma_k))'(x_k) / |(H \circ (\text{id} \otimes \gamma_k))'(x_k)|$$

and $(H \circ (\text{id} \otimes \gamma_k))'(x_k) = (1, (h \circ (\text{id} \otimes \gamma_k))'(x_k))$.

We combine (6.22), (6.23), (6.24), (6.25), (6.30), (6.31) and $N \geq 4$ to conclude that

$$\begin{aligned} \angle(\dot{\Gamma}_{i+1}(u_1), \dot{\Gamma}_{i+1}(u_2)) &\leq \pi (|\Delta C_1| + |\Delta C_2| + |\Delta C_3|) \\ &\leq \pi \left(3\delta_0 a_{i+1}^{2/3} + 4 \cdot 10^{2-N} + 10^{7-N} \right) \alpha(u_1, u_2) \leq \delta_0 \left(\frac{|u_1 - u_2|}{2s_{i+1}} \right)^{2/3}, \end{aligned}$$

where the last inequality holds provided that $N = N(\delta_0)$ is chosen large enough. This completes the proof of Lemma 6.16. \square

Lemma 6.17. *Condition (C) of Definition 6.10 is satisfied, and $\|w_{i+1}\|_q \leq 2\|w_i\|_q$ on Q .*

Proof. Let us write $\|D_{\dot{\Gamma}_i} H(v)\|$ for the magnitude of the directional derivative of H in the direction of $\dot{\Gamma}_i$ at the point v . Since H is C^2 on an open set,

$$w_{i+1}(H(v))JH(v) = \|D_{\dot{\Gamma}_i} H(v)\|w_i(v)$$

for every v in the domain of H ; here JH denotes the Jacobian of H . Thus, for $u \in \text{spt } \Gamma_{i+1}$,

$$(6.32) \quad w_{i+1}(u) = JH^{-1}(u) \|D_{\dot{\Gamma}_i} H(H^{-1}(u))\| w_i(H^{-1}(u)).$$

We want to show that, on any given ball of radius s_{i+1} , there is a constant A' so that w_{i+1} takes values in $[(1 + \delta_0)^{-1}A', (1 + \delta_0)A']$ and is $\frac{2}{3}$ -Hölder continuous with constant $\frac{A'\delta_0}{(2s_{i+1})^{2/3}}$.

Sublemma 6.18. *On any ball of radius s_{i+1} , JH^{-1} is $10^{5-2N}s_{i+1}^{-1}$ -Lipschitz with values in $[(1 + 10^{3-N})^{-1}, 1 + 10^{3-N}]$. In particular, JH^{-1} is $\frac{2}{3}$ -Hölder continuous with constant $\frac{10^{6-2N}}{(2s_{i+1})^{2/3}}$.*

Proof. First, we compute the differential of H :

$$DH(x, y) = \begin{pmatrix} 1 & 0 \\ \tilde{\varphi}'(x)g(x, y) + \tilde{\varphi}(x)g_x(x, y) & 1 + \tilde{\varphi}(x)g_y(x, y) \end{pmatrix}.$$

Thus

$$(6.33) \quad JH = 1 + \tilde{\varphi}(x)g_y(x, y).$$

Outside Q , near the edge of Q , or near γ_0 , we have $JH \equiv 1$, so $JH^{-1} \equiv 1$. It remains to consider the case when v_1, v_2 are in Q and above γ_0 . We see that

$$g_y(x, y) = \frac{1}{10^N s_{i+1} - \gamma_0(x)} \varphi' \left(\frac{y - \gamma_0(x)}{10^N s_{i+1} - \gamma_0(x)} \right).$$

Now $10^N s_{i+1} - \gamma_0$ is 10^{-2} -Lipschitz and takes values in $[0.99 \cdot 10^N s_{i+1}, 1.01 \cdot 10^N s_{i+1}]$, thus $(10^N s_{i+1} - \gamma_0)^{-1}$ is $10^{-1-2N}s_{i+1}^{-2}$ -Lipschitz and takes values in $[0.98 \cdot 10^{-N}s_{i+1}^{-1}, 1.02 \cdot 10^{-N}s_{i+1}^{-1}]$. On the other hand, $\varphi'(\frac{y - \gamma_0(x)}{10^N s_{i+1} - \gamma_0(x)})$ has size at most $\frac{5}{2}$ and is Lipschitz with constant

$$\|\varphi''\|_\infty (1.01 \cdot 10^N s_{i+1} \cdot 10^{-1-2N}s_{i+1}^{-2} + 1.03 \cdot 10^{-N}s_{i+1}^{-1}) \leq 10^{2-N}s_{i+1}^{-1}.$$

Thus g_y has size at most $10^{1-N}s_{i+1}^{-1}$ and is Lipschitz with constant

$$1.02 \cdot 10^{-N}s_{i+1}^{-1} \cdot 10^{2-N}s_{i+1}^{-1} + \frac{5}{2} \cdot 10^{-1-2N}s_{i+1}^{-2} \leq 10^{3-2N}s_{i+1}^{-2}.$$

Therefore, JH takes values in $[(1 + 10^{3-N})^{-1}, 1 + 10^{3-N}]$ and is Lipschitz with constant

$$6s_{i+1} \cdot 10^{3-2N}s_{i+1}^{-2} + 10^{1-N}s_{i+1}^{-1} \cdot 10^{2-N} \leq 10^{4-2N}s_{i+1}^{-1}.$$

Since H^{-1} is 1.01-Lipschitz, $JH^{-1} = (JH \circ H^{-1})^{-1}$ takes values in $[(1 + 10^{3-N})^{-1}, 1 + 10^{3-N}]$ and is Lipschitz with constant $10^{5-2N}s_{i+1}^{-1}$. \square

Sublemma 6.19. *On any ball of radius s_{i+1} , $\|D_{\dot{\Gamma}_i} H \circ H^{-1}\|$ is $\frac{2}{3}$ -Hölder continuous with constant $\frac{1}{2}\delta_0(2s_{i+1})^{-2/3}$ and takes values in $[1/1.01, 1.01]$.*

Proof. Writing $v_k = (x_k, \gamma_k(x_k))$ for $\gamma_k \in \Gamma_i$, we calculate

$$\|D_{\Gamma_i} H(v_k)\| = \frac{\|DH_{v_k}(1, \gamma'_k(x_k))\|}{\|(1, \gamma'_k(x_k))\|} = \sqrt{\frac{1 + ((h \circ (\text{id} \otimes \gamma_k))'(x_k))^2}{1 + (\gamma'_k(x_k))^2}}.$$

Thus $\|D_{\Gamma_i} H(v_k)\| \in [1/1.01, 1.01]$. Note that H^{-1} is 1.01-Lipschitz, and that near $z = 1$, \sqrt{z} is 1-Lipschitz. Thus it suffices to show that

$$(6.34) \quad \frac{1 + ((h \circ (\text{id} \otimes \gamma_k))')^2}{1 + (\gamma'_k)^2} \text{ is } \frac{2}{3}\text{-H\"older continuous}$$

with H\"older constant $\frac{1}{3}\delta_0(2s_{i+1})^{-2/3}$. To this end, consider the equality

$$(6.35) \quad \frac{1 + L_1^2}{1 + M_1^2} - \frac{1 + L_2^2}{1 + M_2^2} = \frac{(1 + M_1^2)(L_1^2 - L_2^2) + (1 + L_1^2)(M_2^2 - M_1^2)}{(1 + M_1^2)(1 + M_2^2)}.$$

In our case, $M_k = \gamma'_k$ is at most $\frac{1}{200}$ and is $\frac{2}{3}$ -H\"older continuous with constant $3\delta_0 a_{i+1}^{2/3} (2s_{i+1})^{-2/3}$, while $L_k = (h \circ (\text{id} \otimes \gamma_k))'$ is at most $\frac{1}{100}$ and is $\frac{2}{3}$ -H\"older continuous with constant $\frac{1}{10}\delta_0(2s_{i+1})^{-2/3}$. (This follows from condition (B), for sufficiently large N .) In view of (6.35), (6.34) is satisfied with H\"older constant

$$\frac{1}{10}\delta_0(2s_{i+1})^{-2/3} + 3\delta_0 a_{i+1}^{2/3} (2s_{i+1})^{-2/3} \leq \frac{1}{3}\delta_0(2s_{i+1})^{-2/3}. \quad \square$$

Sublemmas 6.18 and 6.19 combine with (6.32) to show that $\|w_{i+1}\|_q \leq 2\|w_i\|_q$ on Q .

Sublemma 6.20. *On a ball B of radius s_{i+1} , $A^{-1}w_i \circ H^{-1}$ is $\frac{2}{3}$ -H\"older continuous with constant $\frac{10^{-N/2}\delta_0}{(2s_{i+1})^{2/3}}$ and takes values in $[(1 + \delta_0)^{-1}, 1 + \delta_0]$, where A is the constant from condition (C) for B .*

Proof. We already know that $A^{-1}w_i$ is $\frac{2}{3}$ -H\"older continuous with constant $\frac{\delta_0}{(2s_i)^{2/3}}$ and takes values in $[(1 + \delta_0)^{-1}, 1 + \delta_0]$. Since H^{-1} is 1.01-Lipschitz, we conclude that $A^{-1}w_i \circ H^{-1}$ is $\frac{2}{3}$ -H\"older with constant

$$1.01^{2/3} \frac{\delta_0}{(2s_i)^{2/3}} \leq \frac{1.01 a_{i+1}^{2/3} \delta_0}{(2s_{i+1})^{2/3}} \leq \frac{10^{-N/2} \delta_0}{(2s_{i+1})^{2/3}}. \quad \square$$

The following lemma is trivial.

Sublemma 6.21. *If $f_i : X \rightarrow [M_i^{-1}, M_i]$ is α -H\"older continuous with constant L_i for $i \in \{1, 2, 3\}$, then $g : X \rightarrow [(M_1 M_2 M_3)^{-1}, M_1 M_2 M_3]$ given by $g = f_1 f_2 f_3$ is α -H\"older continuous with constant $L_1 M_2 M_3 + M_1 L_2 M_3 + M_1 M_2 L_3$.*

Sublemmas 6.18 to 6.21 together with (6.32) combine to show that $A^{-1}w_{i+1}$ takes values in $[1/1.02, 1.02]$ and is $\frac{2}{3}$ -H\"older continuous with constant at most

$$(2s_{i+1})^{-2/3} \left(10^{6-2N} (1.01)(1 + \delta_0) + (1 + 10^{3-N}) \frac{\delta_0}{2} (1 + \delta_0) + (1 + 10^{3-N})(1.01)10^{-N/2} \delta_0 \right),$$

which is bounded above by $\frac{3}{4}\delta_0(2s_{i+1})^{-2/3}$ provided N is large enough.

Thus on balls of radius s_{i+1} , the ratio of maximum to minimum values of w_{i+1} (which is the same as the ratio for $A^{-1}w_{i+1}$) is at most

$$\left(\frac{1}{1.02}\right)^{-1} \left(\frac{1}{1.02} + \frac{3}{4}\delta_0\right) = 1 + 1.02\frac{3}{4}\delta_0 \leq (1 + \delta_0)^2.$$

We can choose $A' \in [A/1.02, 1.02A]$ as appropriate for the given ball to conclude that the Hölder constant for w_{i+1} on the ball is at most $A\frac{3}{4}\delta_0(2s_{i+1})^{-2/3} \leq \frac{A'\delta_0}{(2s_{i+1})^{2/3}}$. This completes the proof of Lemma 6.16.

Conditions (A), (B) and (C) having been verified for the new measured curve family $(\Gamma_{i+1}, \sigma_{i+1})$, the proof of Proposition 6.14 is now complete. \square

7. UNIFORMIZATION BY ROUND CARPETS AND SLIT CARPETS

In this section we prove Corollaries 1.9 and 1.10 from the introduction, on the existence of round and slit carpets supporting Poincaré inequalities.

The proof of Corollary 1.9 relies on the following uniformization theorem of Bonk [3, Theorem 1.1 and Corollary 1.2].

Theorem 7.1 (Bonk). *Let $\{D_i : i \in I\}$ be a family of pairwise disjoint domains in \mathbb{R}^2 with Jordan curve boundaries $S_i = \partial D_i$. Assume that the curves $\{S_i\}$ are uniformly relatively separated uniform quasicircles. Then there exists a quasiconformal map $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $f(D_i)$ is a disc for all $i \in I$. In particular, if T is a planar carpet whose peripheral circles are uniformly relatively separated uniform quasicircles, then T can be mapped to a round carpet T' by a quasiasymmetric homeomorphism f . Furthermore, if T has measure zero, then f is unique up to post-composition with a Möbius transformation.*

A Jordan curve $S \subset \mathbb{R}^2$ is a *quasicircle* if it is the image of \mathbb{S}^1 under a quasiconformal map of \mathbb{R}^2 . A family of Jordan curves $\{S_i\}_{i \in I}$ consists of *uniform quasicircles* if there exists $K \geq 1$ so that each S_i is the image of \mathbb{S}^1 under a K -quasiconformal map of \mathbb{R}^2 . Finally, a family of Jordan curves $\{S_i\}_{i \in I}$ is *uniformly relatively separated* if there exists $c > 0$ so that

$$(7.1) \quad \frac{\text{dist}(S_i, S_j)}{\min\{\text{diam } S_i, \text{diam } S_j\}} \geq c \quad \forall i, j \in I, i \neq j.$$

Theorem 7.1 is stated in [3] for the extended complex plane $\widehat{\mathbb{C}}$ endowed with the spherical metric. However, an application of the (conformal) stereographic projection mapping converts the statement in [3] to our formulation.

7.1. Quasiasymmetric uniformizability of $S_{\mathbf{a}}$ by round carpets. In this subsection we prove Corollary 1.9.

For any \mathbf{a} , the peripheral circles of the carpet $S_{\mathbf{a}}$ are uniformly separated. Indeed, when $\mathbf{a} \in c_0$, the uniform relative separation condition (7.1) holds in the following stronger form:

$$(7.2) \quad \lim_{\max\{\text{diam } S_i, \text{diam } S_j\} \rightarrow 0} \frac{\text{dist}(S_i, S_j)}{\min\{\text{diam } S_i, \text{diam } S_j\}} \rightarrow \infty.$$

Since the peripheral circles are rigid squares, they are uniform quasicircles. By Theorem 7.1, $S_{\mathbf{a}}$ is quasiasymmetrically uniformized by a round carpet T' whenever $\mathbf{a} \in c_0$.

When $\mathbf{a} \notin \ell^2$ (and so $S_{\mathbf{a}}$ has zero Lebesgue measure), T' is rigid, i.e., unique up to the application of a Möbius transformation.

When $\mathbf{a} \in \ell^2$ (and so $S_{\mathbf{a}}$ has positive Lebesgue measure), we observe that $T' = f(S_{\mathbf{a}})$ is an Ahlfors 2-regular subset of \mathbb{R}^2 (the volume upper bound is trivial; the lower bound follows from the quasisymmetry of f as in Corollary 3.10 and Remark 3.6(1) of [33]). It now follows from Theorem 1.6 and a result of Koskela and MacManus [27, Theorem 2.3] that the round carpet T' supports a p -Poincaré inequality for some $p < 2$.

7.2. Quasisymmetric uniformizability of $S_{\mathbf{a}}$ by slit carpets. We now turn to the proof of Corollary 1.10.

Let $S_{\mathbf{a}}$ be a carpet with $\mathbf{a} \in \ell^2$. For each m , the interior $S_{\mathbf{a},m}^o$ of the precarpet $S_{\mathbf{a},m}$ is a finitely connected domain in the plane. By Koebe's uniformization theorem (see for instance [14, V§2]), $S_{\mathbf{a},m}^o$ can be conformally uniformized to a parallel slit domain D_m . Now the identity map between the Euclidean metric and the internal metric δ on D_m is conformal, hence $S_{\mathbf{a},m}^o$ is conformally equivalent to (D_m, δ) .

By Proposition 3.1 and the proof of Theorem 1.6, the precarpets $S_{\mathbf{a},m}$ are Ahlfors 2-regular and support a p -Poincaré inequality for any $p > 1$ with constants independent of m . In particular, such precarpets are 2-Loewner in the Euclidean metric; since they are quasiconvex (with constant independent of m) they are also 2-Loewner in the internal metric.

It is straightforward to check that the domains (D_m, δ) are LLC. We now appeal to a theorem of Heinonen [17, Theorem 6.1] which asserts that any quasiconformal map from a bounded domain which is Loewner in the internal metric to a bounded LLC domain is quasisymmetric. The preceding result is quantitative in the usual sense; in our situation this implies that the relevant quasisymmetric distortion function is independent of m . Moreover, the target domains (D_m, δ) are uniformly Ahlfors 2-regular by an argument similar to that used in subsection 7.1.

Passing to the limit as $m \rightarrow \infty$, we obtain a quasisymmetric map from $S_{\mathbf{a}}$ onto an Ahlfors 2-regular parallel slit carpet. To complete the proof, we again use [27, Theorem 2.3] to conclude that the target carpet supports a p -Poincaré inequality for some $p < 2$.

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